Problem Sheet 4: Due Thurs 8th April

Vertically propagating stationary Rossby waves.

Here you will analyse stationary Rossby wave dynamics in the 3D QG system. In this system potential vorticity conservation may be written

$$\frac{\partial q}{\partial t} + \vec{v}.\nabla q = \dot{q} \tag{1}$$

where

$$q = \beta y + \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o \frac{f_o^2}{N^2}\frac{\partial}{\partial z^*}\right)\right]\psi' \tag{2}$$

and

$$\vec{v} = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) \tag{3}$$

with ψ being the stream function that is related to the geopotential $\psi = \Phi/f_o$. In 1 a PV source has been introduced.

In this assignment you will find, numerically, the stationary response of a zonal flow to a small amplitude sinusoidal perturbation at the boundary. The linearised problem has the form

$$U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = \dot{q}', \quad 0 \leq z \leq \infty, \quad -\infty \leq x \leq \infty, \quad -\infty \leq y \leq \infty \tag{4}$$

where U and Q are the basic state zonal wind and PV and the primed quantities are the zonally varying perturbations. The lower boundary condition is

$$\psi' = C_o cos(kx) \text{ at } z = 0 \tag{5}$$

where C_o is a constant.

(a) Verify, for $\dot{q}' = 0$, $\rho_o(z) = exp(-z/H)$ and U and N^2 positive constants, that a solution to (4) to (5) is $\psi' = C_o exp(z/2H) cos(kx + mz)$, where

$$m = \frac{N}{f_o}\sqrt{\frac{\beta}{U} - k^2 - \left(\frac{f_o}{2NH}\right)^2} \tag{6}$$

(b) Calculate the vertical group velocity. A solution with $m \to -m$ would also satisfy the equations. Why in 6 has only the positive value of m been chosen?

(Note that although this solution grows exponentially with height, the perturbation energy is proportional to $\rho_o|\psi'|^2$, which remains bounded as $z \to \infty$).

(c) In 4, $\dot{q}'$ represents diabatic heating that acts only on the wave and not on the basic state. In the atmosphere, and particularly in the stratosphere, thermal fluctuations are damped by longwave radiation in a manner well approximated by Newtonian damping acting on what is called a "radiative damping time" τ, with τ typically being three weeks or more. Under QG scaling, we write the thermal damping term as

$$\dot{q}' = -\frac{1}{\tau\rho_o}\frac{\partial}{\partial z}\left(\frac{\rho_o f_o^2}{N^2}\frac{\partial\psi'}{\partial z}\right) \tag{7}$$

We seek solutions of the form $\psi' = Re\left[C(z)exp(ikx)\right]$. Show that (4) to (7) becomes, for these solutions,

$$\left(U - \frac{i}{\tau k}\right)\frac{1}{\rho_o}\left(\frac{\rho_o f_o^2}{N^2}C'\right)' + \left(\frac{\partial Q}{\partial y} - k^2 U\right)C = 0 \tag{8}$$

where the primed quantities denote ordinary derivatives with respect to z.

(d) The script *rossby_wave_vert.py* will solve these equations numerically for the general case where U and hence $\partial Q/\partial y$ are z dependent. The equations are discretized over a finite vertical domain $0 \leq z \leq H_d$ where the top of the domain $H_d >> H$, where H is the density scale height. We divide the domain up into N layers of thickness $\Delta = H_d/N$. It is assumed that $C(z = 0) = C_o$ as per the lower boundary condition. The value of C at $z = H_d$ is also set to zero but for sufficiently large H_d the exact value of $C(H_d)$ is unimportant.

The script solves the finite difference problem of the form

$$L_{ij}C_j = F_i \to C_i = L_{ij}^{-1}F_j, \text{where } i, j = 1, ..., N \tag{9}$$

Here $C_j = C(z_j)$ and the components of F_i are only terms related to the boundary conditions. The finite difference scheme uses the following approximation for the z derivative of a function r

$$r'(z) \approx \frac{r(x + \Delta/2) - r(z - \Delta/2)}{\Delta} \tag{10}$$

This finite difference approximation thus uses data from the half level above and below z.

(1) Demonstrate that you understand how the matrix L_{ij} and the vector F_i are related to the equations the code is solving. Start by writing the finite difference version of 8 at some level $z_j = j\Delta$ and demonstrate how the matrix L_{ij} is related to this. Then look at the cases $j = 1$ and $j = N$ to understand how the boundary conditions are set up.

(2) Run the script for the test case $U = 5ms^{-1}$. In Figure 1 is plotted the finite difference solution and overplotted in red and green symbols are other analytic solutions. Figure out what each of these analytical solutions represents (there are clues in the code). Figure 2 shows contours of the stream function of the wave in the $x - z$ plane. Do you see any evidence of a poleward heat transport? Give reasoning to back up your answer.

(3) Now, investigate the effect of varying some of the parameters and show plots to back up your answers.

- What happens if you put in a critical layer? i.e. at some point in the vertical profile set the zonal wind to zero. Explain what you see in terms of the theory.
- What happens for an easterly wind profile? Based on this would you expect to see vertically propagating planetary waves in the summer or winter stratosphere? Give reasoning.
- Investigate various different zonal wind profiles and the effect of varying the static stability on the wave solutions and explain what you see. There are some clues of variations that may be interesting to examine in the code e.g. you may want to alter the static stability to mimic the troposphere and stratosphere i.e. a region of lower static stability below a region of higher static stability.