Problem Sheet 3: Due Thurs 10th March

Stationary Rossby waves introduced by flow over topography.

1. In the presence of bottom topography (h_B) on a β-plane with a rigid lid, potential vorticity conservation takes the form

$$\frac{Dq}{Dt} = 0 \qquad \text{where} \qquad q = f_o + \beta y + \zeta + \frac{f_o h_B}{H} \tag{1}$$

where ζ is the relative vorticity and H is the mean depth of the shallow water layer. Consider a constant zonal flow (U) passing over y-independent topography whose height h_B is small compared to the mean depth of the fluid.

(a) Show that the **steady**, **small amplitude**, **y-independent** stream function that satisfies this PV conservation satisfies the following equation

$$\frac{d^2\psi'}{dx^2} + \frac{\beta}{U}\psi' = -\frac{f_o h_B}{H} \tag{2}$$

(b) In the following you will solve this equation for the case of bottom topography of the form $h_B = A\delta(x)$ in the domain $-\infty < x < +\infty$. A is a constant with units of length squared and $\delta(x)$ is the dirac delta function i.e. you will find the Green's function for the equation. This topography is such that $\int_{-\infty}^{+\infty} h_B dx = A$ where A may be interpreted as the product of the mean height and the mean width of a typical mountain. (A mountain in the shape of a δ function may seem rather artificial but the Green's function can be used to build solutions for arbitrary topography.) You may follow the following hints to do this:

- Assume that $\psi' = 0$ upstream of the mountain
- Assume that downstream of the mountain the solution satisfies

$$\frac{d^2\psi'}{dx^2} + \frac{\beta}{U}\psi' = 0 \tag{3}$$

  which you should prove has a solution of the form

$$\psi' = 2iBsin\left(\sqrt{\frac{\beta}{U}}x\right) \tag{4}$$

  where B is an arbitrary constant. Note that you must match the solutions upstream and downstream at $x = 0$.

- Solve for B by patching the solutions at $x = 0$ i.e. integrate Eq. 2 over an infinitesimal region ($-\epsilon$ to $+\epsilon$) around the origin $x = 0$.
- You should arrive at the answer

$$\begin{aligned} \psi' &= 0 \text{ for } x < 0 \\ &= -\frac{f_o}{H}\sqrt{\frac{U}{\beta}} A sin\left(\sqrt{\frac{\beta}{U}}x\right) \text{ for } x > 0 \end{aligned} \tag{5}$$

(c) Plot the velocity perturbation (v') and geopotential height perturbation $Z' = f_o\psi'/g$ assuming U=10ms^{-1}, $A = (1km)(200km) = 2 \times 10^8 m^2$, H =10km, f_o=10^{-4}s^{-1} and β =10^{-11}m^{-1}s^{-1}.

2. You will now carry out a numerical calculation that bulids on the solution of the previous question. This problem solves the 1-D stationary Rossby wave problem for flow over a ridge in a periodic domain that corresponds to a latitude circle.

For more realistic situations some damping will act on the velocity fields. This may be considered to take the form of a damping on the relative vorticity such that

$$\frac{D\zeta}{Dt} = -\frac{\zeta}{\tau}$$

where τ is a damping timescale.

(a) Prove that in the presence of such a damping the solution (ψ') now satisfies

$$\frac{d^2\psi'}{dx^2} + \frac{1}{U\tau}\frac{d\psi'}{dx} + \frac{\beta}{U}\psi' = -\frac{f_o h_B}{H} \tag{6}$$

(b) This equation can be solved subject to a periodic boundary condition $\psi'(x) = \psi'(x + 2\pi j a cos(\phi_o))$, where a is the Earth's radius, ϕ_o is a reference latitude, $2\pi a cos\phi_o$ is the circumference of the latitude circle at that latitude, and j is an integer.

The circle is divided into n even intervals of width

$$\Delta x = 2\pi a cos(\phi_o)/n$$

and the finite difference formulas

$$\frac{dg(x_j)}{dx} \approx \frac{g(x_{j+1}) - g(x_{j-1})}{2\Delta x}$$

and

$$\frac{d^2 g(x_j)}{dx^2} \approx \frac{g(x_{j+2}) - 2g(x_j) + g(x_{j-2})}{(2\Delta x)^2}$$

are used to approximate the spatial derivatives. The finite difference problem then takes the form

$$L_{jk}\psi'(x_k) = -\frac{f_o h_b(x_k)}{H} \rightarrow \psi'(x_j) = -\frac{f_o L_{jk}^{-1} h_b(x_k)}{H} \qquad (7)$$

where L_{jk} is an $n \times n$ array, the indices run from 1 to n and summation over repeated indices is implied.

Download the python code *rossby_wave_1d.py* from the website and demonstrate that you understand how the linear operator L_{jk} is created, using the finite difference formulas and the linear equation 6. (The code makes use of a subroutine "*get_index*" which implements the periodic boundary condition. For example

$$\text{when } j = 1, \psi'(x_{j-2}) = \psi'(x_{-1}) = \psi'(x_{n-1}),$$

calling get_index(-1,n) returns the value $n - 1$ in this case).

(c) The default case produces a Rossby wave train in response to a narrow Gaussian ridge located at $x = 0^o$ longitude. The parameters are set for 45N. Run this code. Compare the solution you obtain with the analytical solution found for question 1 (You may want to overplot the analytical solution with this one) (Show plots).

(d) Verify that the wavelength and amplitude dependence predicted by the analytical solution agrees with the numerical solution.

(e) What is the effect of varying the damping?

(f) What is the effect of varying the height of the topography?

(g) What is the effect of varying the width of the topography?

(h) What happens if you use an Easterly wind? Can you explain this?

(Show plots to back up your answers).

(i) An analytical solution that includes damping predicts that, as long as the damping isn't too strong, the oscillatory solution is multiplied by an envelope function $e^{-|x|/(c_g\tau)}$ where $|x|$ is the distance away from the mountain and c_g is the group velocity for the stationary Rossby wave. Find the group velocity for a stationary Rossby wave and see whether this damped analytic solution matches the numerical one.