Problem Sheet 1: Due Thurs 3rd Feb

1. Primitive equations in different coordinate systems

(a) Using Lagrangian considerations and starting from an infinitesimal mass element in cartesian coordinates (x, y, z) make use of hydrostatic balance to demonstrate that the mass continuity equation in pressure coordinates (x, y, p) is given by

$$\frac{\partial u}{\partial x}\Big|_p + \frac{\partial v}{\partial y}\Big|_p + \frac{\partial \omega}{\partial p} = 0, \tag{1}$$

where ω is the pressure velocity (Dp/Dt).

(b) In geometric height coordinates in the rotating frame of the Earth, momentum balance for an inviscid fluid is given by

$$\frac{D\vec{v}}{Dt} + 2(\vec{\Omega} \times \vec{v}) = -\frac{\nabla p}{\rho} - \nabla gz, \tag{2}$$

where g, the acceleration due to gravity, is assumed to be constant. Starting from 2, and assuming hydrodstatic balance transform the vertical coordinate from geometric height to pressure to demonstrate that, in pressure coordinates (x, y, p), momentum balance for the horizontal components of velocity $(\vec{v}_H)$ in an inviscid fluid in the absence of friction is given by

$$\frac{D\vec{v}_H}{Dt} + 2(\vec{\Omega} \times \vec{v}_H) = -\nabla\Phi \tag{3}$$

where Φ is the geopotential at the relevant pressure level.

State the dominant balances for atmospheric motion in the extratropics and demonstrate that the geostrophic velocity in pressure coordinates is non-divergent.

2. Inertial Oscillations

Consider the motion of a free particle on a frictionless parabolic surface. The parabolic surface has a height profile given by $h = h_o + ar^2$ where h_o is the height of the surface at the origin as shown in Figs. 1 (a) and (b) and $r = \sqrt{x^2 + y^2}$ (the displacement of the particle from the origin). A particle starts its trajectory at the origin $(x, y) = (0, 0)$ with velocity components in the x and y directions given by $(u, v) = (0, v_o)$.

(a) Considering momentum balance in the x-y plane find the positions x and y and velocities u and v as a function of time.

(b) Now consider the case depicted in Fig. 1 (c) where this parabolic surface is rotating with angular velocity $\Omega = \sqrt{2ga}$. Calculate the positions x and y and velocities u and v as a function of time as viewed by an observed in the rotating frame. Sketch the trajectory of the particle.

Write down an equation for the stream function of the motion.

This situation is analogous to that of "inertial oscillations" in the Earths atmosphere and oceans. The parabolic dish and the shape of the Earth are analogous. The Earth is an oblate spheroid and so at all latitudes other than the equator and the pole there is actually a small component of gravity in the latitudinal direction. But, this component of gravity is such that it counteracts the centrifugal force preventing bouyant objects from sliding toward the equator. In situations where the pressure gradients are small, the dominant force acting on fluid elements is the coriolis force and inertial oscillations occur. The restoring force for the oscillations is the coriolis force. These are particularly important in the ocean. Circulations always veer to the right in the NH and left in the SH

(c) Demonstrate that in the presence of Rayleigh damping i.e. $F_f =$

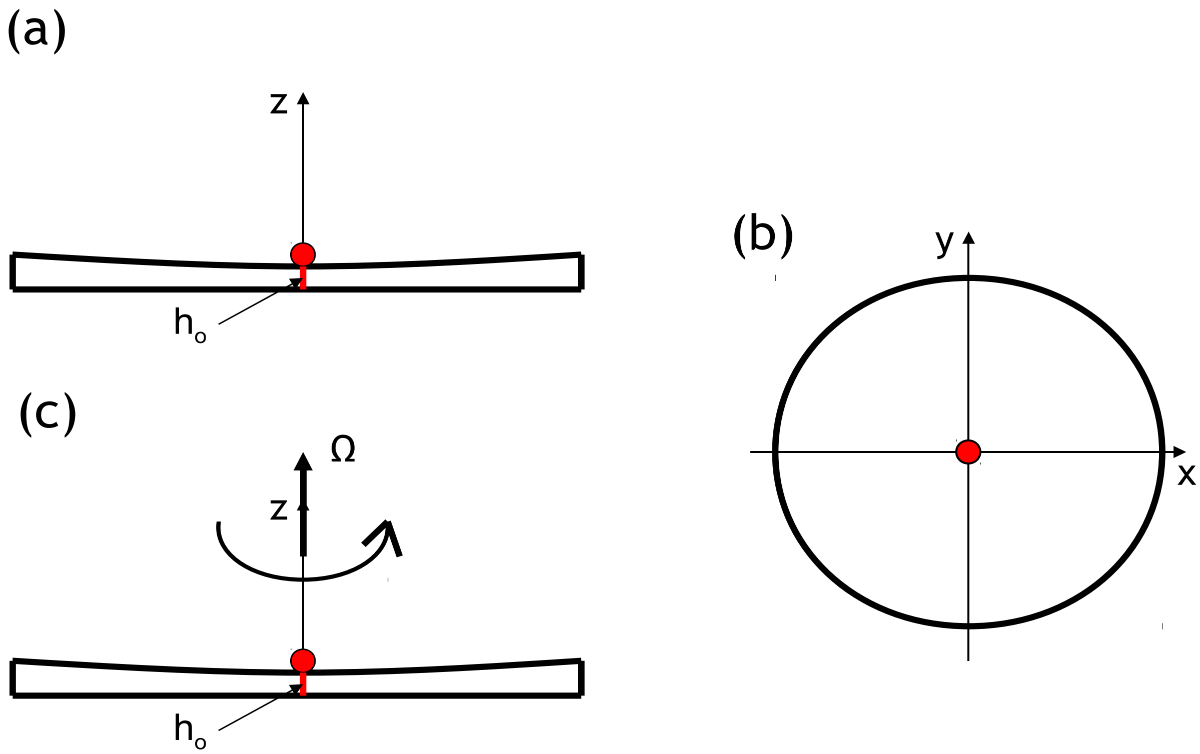

Figure 1: Schematic of the parabolic surface considered in question 2. (a) side on view (b) top-down view and (c) for the case of rotation.

$(-Ku, -Kv)$, where K is a constant frictional coefficient, that the velocity decays exponentially with time.

(d) Now consider the case in part (b) but in the presence of a periodic forcing in the x direction with a frequency ω. The equations of motion in this case are

$$\frac{du}{dt} - fv = Acos\omega t \qquad \frac{dv}{dt} + fu = 0, \tag{4}$$

where $A > 0$ is the amplitude of the external force per unit mass and ω is the driving frequency. Solve for the motion assuming a solution that oscillates with the driving frequency ω where $\omega \neq f$. That is do not consider the inertial oscillations with frequency f which represent the homogeneous part of the solution. We have seen in part (c) that in the presence of damping such solutions decay exponentially with time.

Define an appropriate Rossby number (R_o) for this motion.

Plot the solutions and discuss the balance of the terms in Eq. 4 when 1) $R_o >> 1$, 2) $R_o << 1$.

3. The β plane approximation

The *β-plane approximation* is a tangent plane approximation to the equations of motion on the sphere. Consider the Coriolis parameter $f = 2\Omega \sin\phi$. We define a reference latitude ϕ_0 and Taylor-expand the Coriolis parameter about this reference latitude as follows:

$$f = 2\Omega \sin\phi \approx 2\Omega(\sin\phi_0 + (\phi - \phi_0)\cos(\phi_0)) = f_0 + \beta y, \tag{5}$$

where $f_0 = 2\Omega \sin\phi_0$; $\beta = 2\Omega\cos\phi_0/a$, with a the Earth's radius; and $y = a(\phi - \phi_0)$ is a Cartesian meridional coordinate that represents the displacement from the reference latitude. The equations of motion for the free particle problem now become

$$\frac{du}{dt} - (f_0 + \beta y)v = 0, \ \frac{du}{dt} + (f_0 + \beta y)u = 0, \tag{6}$$

which is a pair of nonlinear ordinary differential equations. In the following these equations will be integrated numerically.

Read through and try out the Python resources at compwiki.physics.utoronto.ca, focussing on the tutorials. Then run the script coriolis_on_beta.py which is available on the course website. This script integrates the equations of motion, plots the trajectory, and saves a hardcopy of the output.

(a) Run the script with the value of f_o corresponding to 30^o latitude and $\beta = 0$ and verify the calculation reproduces the solution for the Problem 2(b). Take v_o =1ms^{-1}. Hand in an annotated copy of the script that demonstrates your understanding of it.

(b) Now set the values of β and f_o for $\phi_o = 30^o$N latitude. What is the main change introduced by turning on β? Provide a qualitative explanation for what you see.

(c) Investigate the tropical regime by setting ϕ_o =0. Run the script and document the results for various initial directions and speeds.

4. Checking on scaling assumptions and geostrophic balance.

Fig. 2 is a recent analysis chart of 500mb fields over North America. In the following problem we will be concerned with the black contours which show the 500mb geopotential height (in units of 10m) and the wind barbs, which indicate the direction and wind speed (A google search on wind barbs will provide the wind barb conventions).

Annotate the map indicating some regions where geostrophic balance holds well and regions where geostrophic balance does not seem to hold.

Choose a region where geostrophic balance seems to hold and estimate the magnitude of the geostrophic wind you would expect given the geopotential height variations. How does it compare with the wind observations?

Provide an order of magnitude estimate of the Rossby number for this location. Would you expect geostrophic balance to hold?

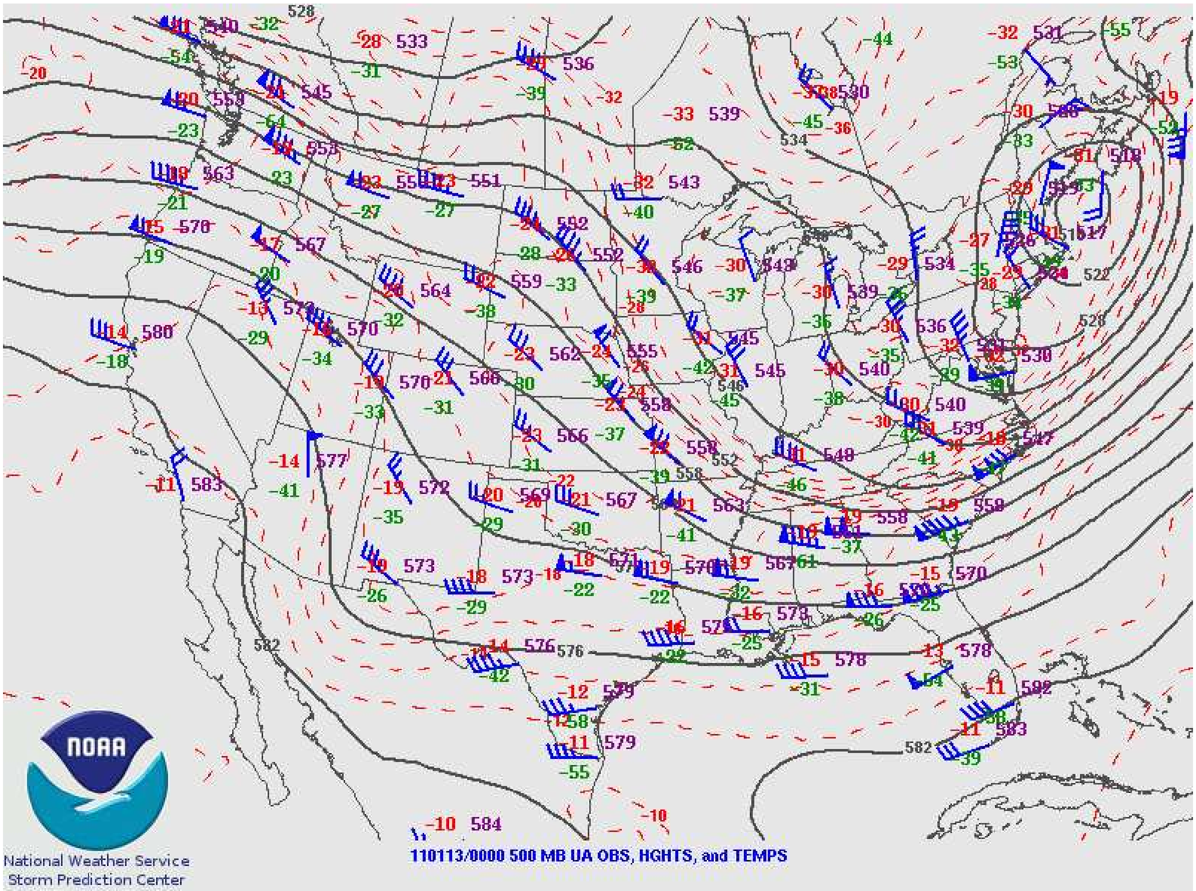

Figure 2: Recent analysis of 500mb fields over North America from http://www.spc.noaa.gov/obswx/maps. Black contours show geopotential height (in 10m units) and wind barbs indicate wind direction and speed.