8 Baroclinic Instability

The previous sections have examined in detail the dynamics behind quas-geostrophic motions in the atmosphere. That is motions that are of large enough scale and long enough timescale for rotation to be important and for the Rossby number to be small. The dynamics of Rossby waves has been examined. These are wave motions that require a background gradient of potential vorticity for their propagation. The mechanism of production of large scale Rossby waves by flow over topography has been discussed together with the propagation of such waves vertically into the stratosphere. This section will now focus on the mechanism responsible for the smaller scale baroclinic eddies that are ubiquitous throughout the mid-latitude troposphere. In understanding the process that leads to these motions we will also gain an understanding of why the atmosphere is full of synoptic scale motions of a similar size. There are typically 5 or 6 synoptical scale eddies around a latitude circle at any one time. These eddies have characteristic length scales close to the Rossby radius of deformation L_D.

These eddies are also quasi-geostrophic motions that grow by feeding on the available potential energy associated with the meridional temperature gradient in mid-latitudes - the process of **Baroclinic Instability**.

8.1 Introduction

Baroclinic instability is relevant in situations where there is

- Rotation
- Stratification
- A horizontal temperature gradient

Consider the process of geostrophic adjustment examined in Section 4.4. An initially unbalanced situation consisting of a step-function in the depth of a shallow water layer is allowed to adjust to equilibrium, first in the absence of rotation and then with a constant rotation f_o. In the absence of rotation the shallow water layer equilibrates to a situation with zero height anomaly. All the initial available potential energy is converted to kinetic energy. In contrast, the presence of rotation inhibits the adjustment of the fluid. Adjustment only occurs over a small region around the initial step function until a geostrophically balanced state is reached. The presence of rotation inhibits the conversion of potential energy to kinetic energy. As a result, the geostrophically balanced state still has potential energy associated with it.

The geostrophically balanced state in the simpler shallow water system can be seen to be analogous to the situation that occurs in the mid-latitudes. The differential solar heating between the equator and poles means that the tropics are warmer and the poles are cooler. Therefore there is a negative gradient between the equator and poles as can be seen on pressure surfaces in Fig. 1 (a). When a horizontal temperature gradient occurs on a pressure surface the flow is baroclinic and there are vertical wind shears through thermal wind balance (See Fig. 1 (b)).

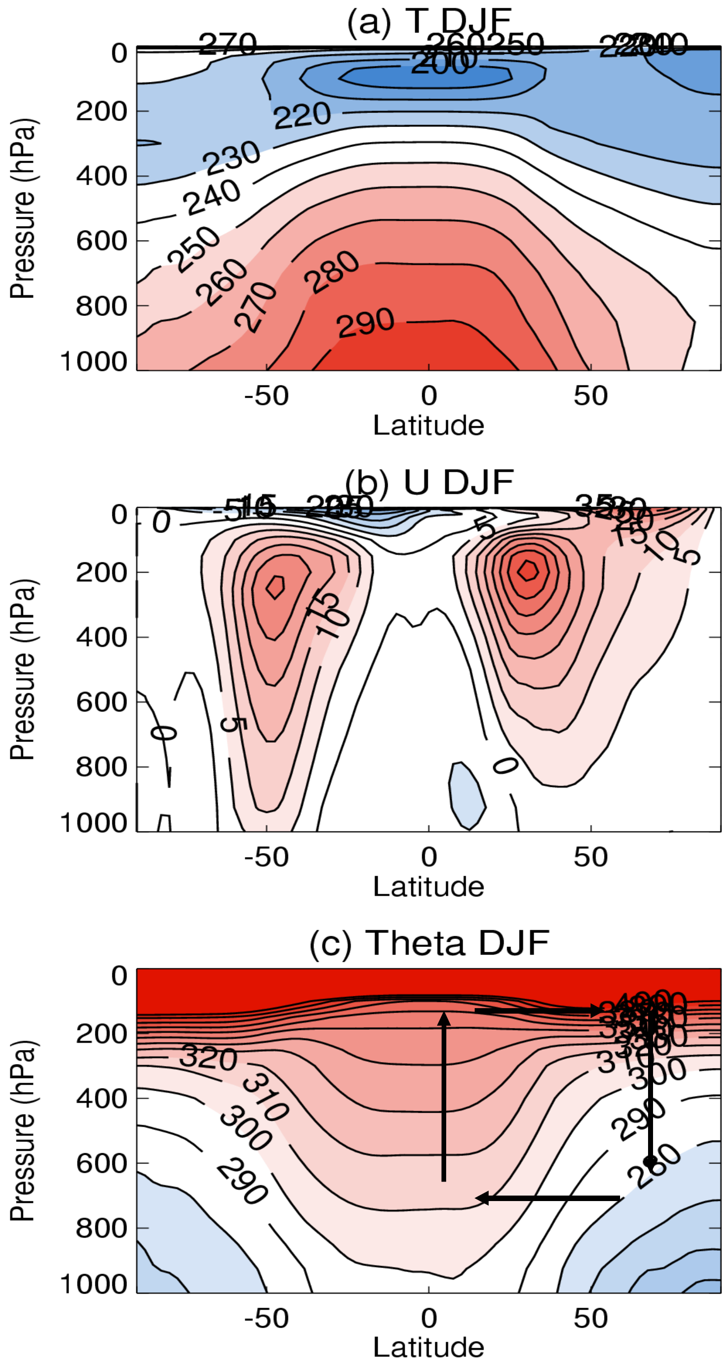

Figure 1: (a) Climatology of zonal mean temperature in DJF on pressure levels from ERA-40. (b) as (a) but zonal mean zonal wind. (c) as (a) but potential temperature. The arrows illustrate, schematically, the overturning circulation that would be set up in the absence of rotation. This would act to flatten the potential temperature contours and reduce the baroclinicity in mid-latitudes.

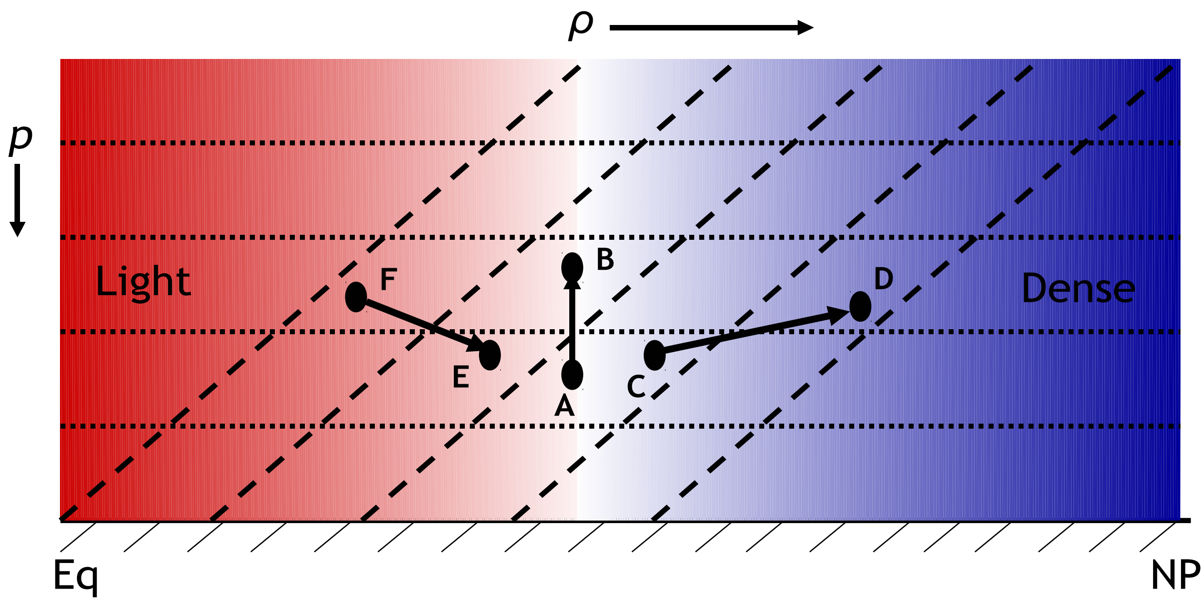

Figure 2: Schematic illustration of sloping convection. Surfaces of constant pressure are illustrated by the dotted lines and surfaces of constant density by the dashed lines.

Now, remembering the relationship between the geopotential of a pressure surface and the integral of tempreature below that surface:

$$\Phi(p) = \Phi(p_s) + \int_p^{p_s} RTdlnp'$$

it can be seen that a particular pressure surface in the tropics will exist at higher geometric height than that same pressure surface at high latitudes i.e. there is a horizontal pressure gradient on geometric height surfaces. This is analogous to the shallow water geostrophically adjusted situation above. So, the situation of an equator to pole temperature gradient on the rotating Earth has potential energy associated with it. Potential energy that may be extracted by the baroclinically growing disturbances.

If the Earth was not rotating this situation would not be maintained. An equator to pole overturning circulation would be set up as depicted in Fig. 1 (c). Potential temperature, being a materially conserved quantity, would be advected by this overturning circulation and it can be seen that this would act to flatten the potential temperature gradients in Fig. 1 (c) until eventually potential temperature (and so temperature itself) is constant on pressure surfaces. In this situation there would no longer be any potential energy available and there would no longer be baroclinicity.

The energetics behind Baroclinic Instability can be illustrated through Fig. 2 This depicts the situation of a warm equator and a cold pole, with pressure surfaces illustrated by the dotted lines and surfaces of constant density depicted by the dashed lines.

The warm air, by the ideal gas law, is lighter and the cold air is denser. Consider the exchange of the parcels between A and B. The denser parcel A is displaced upward and is now surrounded by air of lower density and vice-versa for B. These air parcels will therefore experience a restoring force back to their original positions via the stratification. In contrast, parcels C and D can be exchanged and now the light parcel is surrounded by even denser fluid and so is buoyant and may continue and vice versa for the heavier parcel. Moreover, the center of gravity has been lowered since the heaver parcel has moved lower and the lighter parcel has moved higher. Potential energy has been extracted from the system and converted to kinetic energy of the air parcels.

In the final situation, the exchange of parcels E and F, although the parcels are buoyant relative to their surroundings in their new positions, there has been no lowering of the centre of gravity of the system. The opposite has happened, the lighter parcel is lower and the heavier parcel is higher. So, this motion requires an input of energy and so will not happen spontaneously.

Convection in the sloping sense of the C and D parcels is both allowed by the buoyancy considerations and also releases potential energy and so it's possible that such motions can occur spontaneously and will continue to grow. The motion however, cannot increase indefinately because the effects of rotation will inhibit the poleward motion and induce a circulation.

8.2 The Eady Model

In order to understand the process of baroclinic instability, the Eady model will be used. This was formulated by Eady in 1949. The situation considered is depicted in Fig. 3 and can be summarized as follows.

- The motion is on an f-plane.
- The stratification is uniform i.e. N^2 is constant. This is a reasonable approximation in the troposphere.
- The atmosphere is Boussinesq i.e. density variations are ignored except in the static stability.
- The motion is between two, flat, rigid horizontal surfaces. The upper surface may be considered to be the tropopause with the increase in static stability inhibits vertical motion.
- log-pressure coordinates are used.
- There is a uniform vertical wind shear $u_o = \Lambda z$. By thermal wind balance this must be associated with a horizontal temperature gradient.

The quasi-geostrophic PV equation that will govern the flow is

$$\frac{Dq}{Dt} = 0 \qquad q = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{f_o^2}{N^2}\frac{\partial^2}{\partial z^2}\right]\psi \tag{1}$$

This is the same as the Q-G PV equation derived in Section 7 but since the situation is on an f-plane the term associated with the coriolis parameter can be omitted and given that the situation is considered to be Boussinesq the density terms have cancelled.

This may be written in terms of the Rossby radius of deformation for a stratified atmosphere $L_D = NH/f_o$ as

$$q = \nabla^2\psi + \frac{H^2}{L_D^2}\frac{\partial^2\psi}{\partial z^2} \tag{2}$$

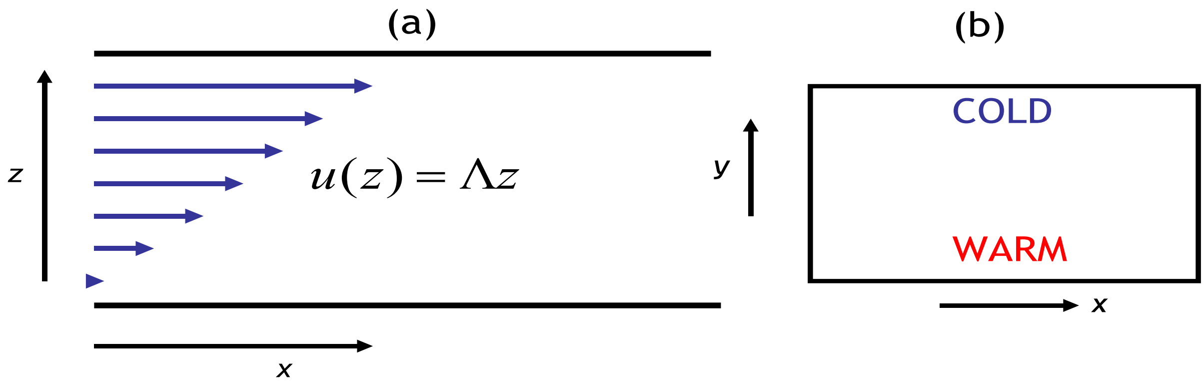

Figure 3: Setup of the Eady model (a) in the x-z plane and (b) in the horizontal.

The background uniform zonal wind shear means the basic state stream function can be written as $\overline{\psi} = -\Lambda yz$. Therefore

$$\overline{q} = \nabla^2\overline{\psi} + \frac{H^2}{L_D^2}\frac{\partial^2\overline{\psi}}{\partial z}^2 = 0$$

So, the potential vorticity of air parcels is initially zero and so it must remain zero for all time.

Note that in this situation there is no variation of the coriolis parameter with latitude and so there is no background gradient of potential vorticity. Therefore, Rossby wave motion cannot simply be induced by meridional displacement of air parcels in the interior as discussed in Section 4.2.4 because there is no background potential vorticity gradient to provide the restoring force. (Note that the more complex situation where β is included is formulated in the Charney model not discussed here.)

The wave motion in the Eady model comes from the boundary conditions.

8.3 Boundary conditions: Eady Edge waves

While there is no background potential vorticity gradient, wave motion is possible at the boundary. Wave motions can exist on a rigid boundary in the presence of a temperature gradient along that boundary.

The boundary condition at $z = 0$ and $z = H$ is that the vertical velocity w must be zero. Therefore, considering the thermodynamic equation

$$\frac{D_g\theta}{Dt} + w\frac{d\theta_o}{dz} = 0$$

Away from the boundary, fluid parcels will move along isentropic surfaces. The advection of potential temperature by the geostrophic wind will be balanced by the advection associated with the vertical velocity so that potential temperature will be materially conserved following the fluid motion. However, on the boundary there can be no vertical motion and so, in the presence of a horizontal temperature gradient, a horizontal displacement of air parcels along the boundary will result in a potential temperature anomaly on the boundary. This temperature anomaly on the boundary will induce a circulation.

The form of the waves on the boundaries can be found by solving the thermodynamic equation in the absence of a vertical velocity. Remembering that

$$\theta = \frac{H}{R} exp\left(\frac{\kappa z}{H}\right)\frac{\partial \phi}{\partial z}$$

and remembering that the stream function is related to the geopotential by $\phi = f_o\psi$ this gives the boundary condition

$$\frac{D_g}{Dt}\left(\frac{\partial \psi}{\partial z}\right) = 0 \qquad \text{at } z = 0 \text{ and } H \tag{3}$$

Consider the stream function to consist of a basic state and a small zonally asymmetric perturbation i.e. $\psi = \overline{\psi} + \psi'$ and linearise to give

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} + v'\frac{\partial}{\partial y}\frac{\partial \overline{\psi}}{\partial z} = 0$$

Remembering that $v' = \partial\psi'/\partial x$, $u_o = \Lambda z$ and $\partial^2\overline{\psi}/\partial y\partial z = -\Lambda$ gives

$$\left(\frac{\partial}{\partial t} + \Lambda z\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} - \frac{\partial \psi'}{\partial x}\Lambda = 0 \tag{4}$$

Searching for wave solutions propagating on the boundary of the form $\psi' = Re(\psi_o)(z)exp(i(kx+ly-\omega t))$ (Note that here we're assuming that the wave is periodic in both the x and y directions. Perhaps a more realistic situation is for the waves to be bounded in the y direction in which case a solution could be of the form $\psi' = \psi_o sin(l(y/Y)2\pi)exp(i(kx-\omega t))$, where Y is the width of the domain in the y direction. In that case the solution would vanish at the lateral boundaries. But, this makes little difference to the key results so the doubly periodic boundary conditions will be used).

Substitution of this into 4

$$(\omega - k\Lambda z)\frac{\partial \psi_o(z)}{\partial z} + k\psi_o(z)\Lambda = 0 \text{ at } z = 0 \text{ and } z = H \tag{5}$$

Considering each boundary seperately and assuming that the other boundary is absent (or is very far away) these boundary conditions can be solved for the amplitude of the wave on each boundary as a function of z. The solution can then be plugged into PV conservation (Eq. 2) to find the dispersion relation for each of the boundary waves.

- **At z =0:** The solution of 4 gives

  $$\psi_o(x) = \psi_o(0)exp\left(-\frac{k\Lambda}{\omega}z\right) \tag{6}$$

  so that the wave solution is given by

  $$\psi'(z) = \psi_o(0)exp\left(-\frac{k\Lambda}{\omega}z\right)exp(i(kx+ly-\omega t)).$$

  On insertion into PV conservation this gives the following dispersion relation

  $$\omega = \frac{k\Lambda f_o}{N(k^2+l^2)^{1/2}}$$

From this it can be seen that the wave will have an Eastward phase speed relative to the mean flow which, at the lower boundary, is zero. Also, the amplitude of the wave decreases exponentially away from the boundary and will have fallen by a factor e at the level where the zonal wind (Λz) is equal to the phase speed (ω/k) of the wave. This level is known as the **steering level**.

- **At z=H:** The solution of 4 gives

$$\psi_o(z) = \psi_o(0) exp\left(-\frac{k\Lambda}{\omega - k\Lambda H} z\right) \tag{7}$$

On substitution of the wave solution into PV conservation the following dispersion relation is obtained

$$\omega = -\frac{k\Lambda f_o}{N(k^2 + l^2)^{1/2}} + k\Lambda H$$

The zonal wind speed at the upper boundary is equal to ΛH and so this represents a wave that has a westward phase speed relative to the mean flow.

So, even though this is a situation where there is no potential vorticity and no background gradient of potential vorticity. The fact that there is a meridional temperature gradient means that on imposing the boundary conditions, a displacement of an air parcel in the meridional direction will induce a circulation on the boundary which will result in an Eastward/Westward propagating wave on the Lower/Upper boundary and the amplitudes of these waves will decay exponentially away from the boundary.

An example of temperature anomalies that could exist by displacement of air parcels on the boundary together with the circulation they induce is depicted in Fig. 4. On the lower boundary a poleward displacement of air parcels will induce a warm anomaly which will be associated with a cyclonic circulation. The opposite is true equatorward displacement.

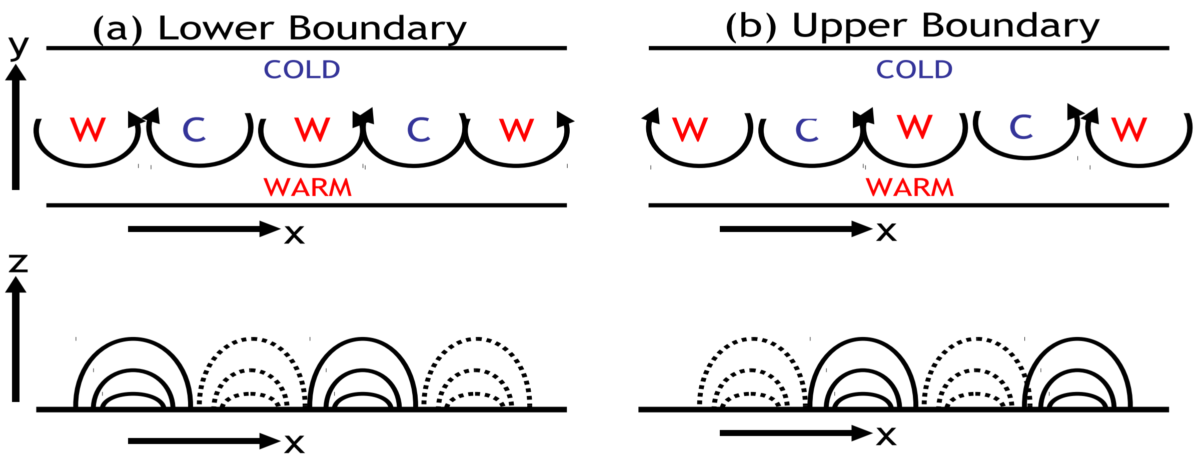

Figure 4: Illustration of the circulation associated with the temperature anomalies on (a) the lower and (b) the upper boundaries. These temperature anomalies would be induced by a meridional displacement of air parcels on the boundaries. The lower panel shows the vertical profile of meridional velocity associated with the temperature anomalies. The solid lines indicate velocity in the positive y direction and the dotted indicated velocity in the negative y direction.

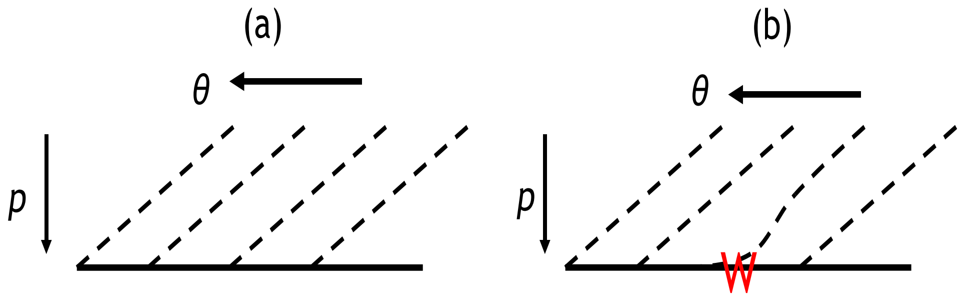

Figure 5: An illustration of the change in potential temperature contours in an infinitesimal region near the boundary, associated with a warm temperature anomaly at the boundary.

It can be seen that the circulalation induced will act to cause the temperature anomalies to propagate toward the East.

One way of thinking about this physically is depicted in Fig. 5. Warming at the surface will act to alter the location of potential temperature surfaces right at the boundary. This will act to increase $|\partial p/\partial \theta|$. By conservation of potential vorticity this must be associated with an increase in the vorticity (i.e. a cyclonic circulation). The opposite occurs at the upper boundary and the wave will propagate to the west.

The lower panels of Fig. 5 show an x-z cross section of the meridional velocity at the boundary. Here, the fall off of the amplitude with height is demonstrated. it can also be seen that there will be no net transport of heat poleward since the v and T anomalies are 90^o out of phase. This edge wave doesn't alter the temperature structure of the atmosphere. It doesn't flatten the potential temperature surfaces or extract energy from the sytem. It is stable.

The opposite occurs at the upper boundary and the wave will propagate to the west. The process of baroclinic instability in the Eady model relies on the presence of these boundary waves and their interaction.

8.4 Solving for unstable growing modes in the Eady problem

The interior PV equation together with the boundary conditions provide 3 equations which can be solved for the wave solutions in the Eady problem. In this section a "normal mode" analysis will be performed to find the most unstable waves. The wave solution is proportional to $exp(-ikct)$. Therefore, in order to have an unstable solution i.e. a solution whose amplitude grows exponentially with time, the phase speed c must be imaginary. In the following, the solutions for which the phase speed is imaginary will be found. Moreover, the solutions that have the largest magnitude of imaginary phase speed will be those that grow fastest and come to dominated the system.

In the interior, potential vorticity can be considered to consist of the basic state potential vorticity and a small perturbation i.e. $q = \overline{q} + q'$. Remembering that $\overline{q} = 0$ everywhere and linearising PV conservation about the basic state gives

$$\frac{\partial q'}{\partial t} + u_o \frac{\partial q'}{\partial x} = 0 \qquad (8)$$

This is the same as the PV conservation equation used to examine Rossby waves in section 4.2.4 but here there is no background gradient of potential vorticity. Remembering that

$u_o = \Lambda z$ and plugging in the expression for the perturbation PV in terms of the stream function gives

$$\left(\frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x}\right)\left(\nabla^2 \psi' + \frac{H^2}{L_D^2}\frac{\partial^2 \psi'}{\partial z^2}\right) = 0 \qquad (9)$$

Again seeking a solution for a wave of horizontal wavenumbers k and l as was done for the boundary conditions ($\psi'(x, y, z, t) = Re(\psi_o)exp(i(kx + ly - \omega t))$ gives

$$(\Lambda z - c)\left(-(k^2 + l^2)\psi_o + \frac{H^2}{L_D^2}\frac{\partial^2 \psi_o}{\partial z^2}\right) = 0 \qquad (10)$$

For $\Lambda z \neq c$ thic gives

$$-(k^2 + l^2)\psi_o + \frac{H^2}{L_D^2}\frac{\partial^2 \psi_o}{\partial z} = 0 \rightarrow \frac{\partial^2 \psi_o}{\partial z^2} - \frac{\mu^2}{H^2}\psi_o = 0 \qquad (11)$$

where this has been written in terms of the parameter $\mu^2 = L_D^2(k^2 + l^2)$. Since a real amplitude is required this equation has solutions of the form

$$\psi_o = Acosh\left(\frac{\mu}{H}z\right) + Bsinh\left(\frac{\mu}{H}z\right) \qquad (12)$$

So this parameter (μ) which depends in the horizontal length scale of the wave determines the vertical structure of the solutions.

This solution can then be inserted into the boundary conditions at $z = 0$ and $z = H$ to give

$$A[\Lambda H] + B[\mu c] = 0$$

$$A[(c - \Lambda H)\mu sinh(\mu) + \Lambda H cosh(\mu)] + B[(c - \Lambda H)\mu cosh(\mu) + \Lambda H sinh(\mu)] = 0$$

These boundary conditions can be written as a matrix equation

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (13)$$

where x_1 and x_2 are the coefficients of A and B in the equation for $z = 0$ and x_3 and x_4 are the coefficients of A and B in the equation for $z = H$.

In order to have non-zero amplitudes A and B, the determinant of the x matrix must be zero. If it is non zero then it's inverse can be formed and the trivial result of $A = B = 0$ is obtained, i.e. there is no wave amplitude.

Forming the determinant of the x matrix and setting it equal to zero gives

$$(\Lambda H)[(c - \Lambda H)\mu cosh(\mu) + \Lambda H sinh(\mu)] - (\mu c)[(c - \Lambda H)\mu sinh(\mu) + \Lambda H cosh(\mu)] = 0$$

which can be re-written as

$$c^2 - \Lambda H c + (\Lambda H)^2(\mu^{-1} coth(\mu) - \mu^{-2}) = 0$$

This is a quadratic equation for the phase speed c which can be solved to give

$$c = \frac{\Lambda H}{2} \pm \frac{\Lambda H}{\mu}\left[\left(\frac{\mu}{2} - coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - tanh\frac{\mu}{2}\right)\right]^{\frac{1}{2}}$$

Exponentially growing solutions require an imaginary phase speed. So, the quantity in the square root must be negative for instability and exponentially growing amplitudes. So we require

$$\left(\frac{\mu}{2} - coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - tanh\frac{\mu}{2}\right) < 0$$

Noting that $\tanh x < x$ for all x the only way to get an imaginary phase speed is to have

$$\frac{\mu}{2} < coth\left(\frac{\mu}{2}\right)$$

This is satisfied if

$$\mu = L_D^2(k^2 + l^2)^{1/2} < \mu_c = 2.399 \tag{14}$$

So, there are only certain horizontal scales of waves that may grow exponentially and the scale of these waves will depend on the rotation rate, the depth of the layer and the static stability which all appear in the expression for the Rossby radius of deformation.