7 Quasi-geostrophic theory for a stratified atmosphere.

We have so far been examining the dynamics of various different types of wave motions in the simplified setting of the shallow water system. This allowed us to understand the dynamics of Rossby waves i.e. waves that are close to geostrophic balance that rely on a background gradient in the potential vorticity for their existence. It was demonstrated that if you formulate the quasi-geostrophic equations (i.e. the equations that effectively filter out the higher frequency noise) then the solutions you are left with are the solutions that correspond to Rossby wave. So, when examining the large scale motion of the atmosphere on the timescale of days to weeks we have to understand the dynamics of Rossby wave propagation.

We then went on to look at the two layer quasi-geostrophic system which was the next step in complexity. The presence of two layers allowed for an additional solution. Two types of Rossby waves could exist in that system: the Barotropic mode and the Baroclinic mode.

In the extratropical atmosphere what we have, for the most part, is a strongly stably stratified atmosphere. This section will therefore go on to examine the quasi-geostrophic theory for a continuosly stratified atmosphere. We therefore now don't make any assumptions about incompressibility and must allow for vertical motion and include the thermodynamic equation in our formulation of Potential vorticity conservation.

We will therefore begin with the full equations for a stratified atmosphere (derived in Section 2) and go through very similar procedures as we did for the shallow water system to derive the Q-G PV equation for a fully stratified atmosphere. Once the Q-G PV equation has been derived it can then be solved to find the Rossby wave solutions for the continuously stratified atmosphere.

7.1 The full primitive equations in log-pressure height coordinates

The full primitive equations in pressure coordinates were derived in Section 2. They are:

$$\left.\frac{\partial u}{\partial x}\right|_p + \left.\frac{\partial v}{\partial y}\right|_p + \frac{\partial \omega}{\partial p} = 0 \tag{1}$$

$$\frac{D\vec{v}_H}{Dt} + f \times \vec{v}_H = -\nabla\Phi \tag{2}$$

$$\frac{D\theta}{Dt} = 0 \tag{3}$$

$$\frac{\partial \Phi}{\partial lnp} = -RT \tag{4}$$

These are mass conservation, momentum conservation, the thermodynamic equation and the final equation is the hypsometric equation which makes use of the ideal gas law to write hydrostatic balance in terms of the geopotential. In deriving the Q-G approximation the vertical coordinate that will be used is log(pressure) height. This is defined as

$$z^* = -Hln\left(\frac{p}{p_s}\right)$$

where H is a scale height given by $H = RT_s/g$, p_S is the surface pressure and T_s is a globally averaged surface temperature. This comes from the combination of hydrostatic balance and the ideal gas law which demonstrates that for an isothermal atmosphere the pressure will fall off exponentially with height (or alternatively the geometric height can be calculated from the the log of the pressure level divided by the surface pressure). The advantage of using z^* instead of p as the vertical coordinate is that the dimensions of the vertical coordinate are now the same as for the horizontal coordinates (i.e. metres rather than pascals). Also we have a vertical coordinate which increases upward unlike pressure which increases downward. For an isothermal atmosphere the z^* will be identical to the geometric height. In the real atmosphere there are temperature variations which result in the scale height ($H = RT_s/g$) being not exactly equal to the true scale height (RT/g) but to a good approximation the scale height H is approximately constant and has a value of about 8km.

So, in an isothermal atmosphere both pressure and density will fall off exponentially with z_*. We may write these pressure and density profiles as

$$p_o = p_s exp(-z^*/H) \qquad\qquad \rho_o = \rho_s exp(-z^*/H)$$

In reality temperature variations will result in slight deviations from these exponential profiles. The primitive equations in pressure coordinates can be derived by transforming 1 to 4 from p to z^* coordinates. Horizontal momentum balance will be unchanged since a balance that occurs on constant pressure surfaces will also ba a balance that occurs on constant log pressure surfaces. The thermodynamic equation will also be unchanged except for the vertical velocity that appears in the material derivativ of potential temperature will now be the vertical velocity in the z^* coordinates (i.e. $w^* = Dz^*/Dt$).

The hypsometric equation (4) can be simply changed into z^* coordinates by noting that $\partial/\partial lnp = -(1/H)\partial/\partial z^*$ giving

$$\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H}.$$

Transformation of the mass conservation equation into z^* coordinates can be made by converting the $\partial\omega/\partial p$ term as follows:

$$\begin{aligned}
\omega = \frac{Dp}{Dt} &= -\frac{p}{H}\frac{Dz^*}{Dt} = -\frac{p}{H}w^* \\
\rightarrow \frac{\partial\omega}{\partial p} = \frac{\partial}{\partial p}\left(-\frac{p}{H}w^*\right) &= -\frac{w^*}{H} - \frac{p}{H}\frac{\partial w^*}{\partial p} = -\frac{w^*}{H} + \frac{\partial w^*}{\partial z^*} \\
&= \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o w^*\right)
\end{aligned}$$

Therefore the mass conservation equation becomes

$$\left.\frac{\partial u}{\partial x}\right|_{z^*} + \left.\frac{\partial v}{\partial y}\right|_{z^*} + \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o w^*\right) = 0$$

So, the starting point for deriving the Q-G equations for the stratified atmosphere are the primitive equations in log(pressure) coordinates as follows

$$\left.\frac{\partial u}{\partial x}\right|_{z^*} + \left.\frac{\partial v}{\partial y}\right|_{z^*} + \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o w^*\right) = 0 \tag{5}$$

$$\frac{D\vec{v}_H}{Dt} + f \times \vec{v}_H = -\nabla\Phi \tag{6}$$

$$\frac{D\theta}{Dt} = 0 \tag{7}$$

$$\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H}. \tag{8}$$

In order to derive the Q-G versions of these equations the same assumptions as in the shallow water system will be used i.e. the Rossby number is small and the velocity components can be separated into geostrophic and ageostrophic contributions

$$u = u_a + u_g \qquad v = v_a + v_g \qquad w^* = w_a^*$$

where $u_a/u_g \sim v_a/v_g \sim R_o$. Note that the vertical velocity only has an ageostrophic component since the geostrophic velocity is purely horizontal.

7.2 Q-G Momentum balance for a stratified atmosphere

The horizontal momentum balance 6 has exactly the same form as the horizontal momentum balance in the shallow water system (Equation 1 in Section 5) with the exception that instead of the pressure gradient being associated with the depth of the fluid layer, here the pressure gradient is associated with the horizontal gradients in geopotential height. But, this is irrelevant for forming the Q-G equations because by definition the horizontal pressure gradients are balanced by the coriolis force on the geostrophic wind so the term on the RHS is only involved in the highest order equation (i.e. the equation for geostrophic balance). Therefore to find the next order equation we need only be concerned with the terms on the left hand side which are identical to the shallow water case. So, the same scaling assumptions as in section 5.2 can be made and exactly the same Q-G horizontal momentum balance equations obtained. These are

$$\boxed{\frac{Du_g}{Dt} - f_o v_a - \beta y v_g = 0} \tag{9}$$

$$\boxed{\frac{Dv_g}{Dt} + f_o u_a + \beta y u_g = 0} \tag{10}$$

7.3 The Q-G thermodynamic equation

The thermodynamic equation can be written in terms of the geostrophic and ageostrophic velocities as follows:

$$\begin{aligned}\frac{D\theta}{Dt} =& \frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = 0\\ \rightarrow& \frac{\partial\theta}{\partial t} + (u_g + u_a)\frac{\partial\theta}{\partial x} + (v_g + v_a)\frac{\partial\theta}{\partial y} + w_a\frac{\partial\theta}{\partial z} = 0\end{aligned}$$

The atmosphere is predominantly stratified in the vertical i.e. the vertical gradients in potential temperature are much greater than horizontal gradients. The potential temperature can therefore be decomposed into a background vertical profile (that would occur for an isothermal atmosphere associated only with the vertica pressure gradient) and a small departure from that

$$\theta = \theta_o + \theta'(x, y, z, t)$$

where

$$\frac{|\partial\theta'/\partial z|}{|\partial\theta_o/\partial z|} \sim O(R_o)$$

The highest order thermodynamic equation is given by

$$\frac{D_g\theta_o}{Dt} = 0 \tag{11}$$

where D_g/Dt represents the material derivative following the geostrophic velocity. The next order equation can be found by scale analysis on the remaining terms as follows (taking the characteristic scale potential temperature perturbations (θ') to be Θ)

$$\begin{array}{ccccccc}
\dfrac{\partial\theta'}{\partial t} & +\,u_g\dfrac{\partial\theta'}{\partial x} & +u_a\dfrac{\partial\theta'}{\partial x} & +\,v_g\dfrac{\partial\theta'}{\partial y} & +v_a\dfrac{\partial\theta'}{\partial y} & +\,w_a\dfrac{\partial\theta_o}{\partial z^*} & +\dfrac{\partial\theta'}{\partial z^*} = 0 \\
\dfrac{\Theta U}{L} & \dfrac{\Theta U}{L} & \dfrac{\Theta U_a}{L} & \dfrac{\Theta U}{L} & \dfrac{\Theta U_a}{L} & \dfrac{W_a\Theta}{R_oH} & \dfrac{W_a\Theta}{H} \\
\dfrac{\Theta U}{L} & \dfrac{\Theta U}{L} & \dfrac{\Theta U_a}{L} & \dfrac{\Theta U}{L} & \dfrac{\Theta U_a}{L} & \dfrac{U_a\Theta}{R_oL} & \dfrac{U_a\Theta}{L} \\
1 & 1 & R_o & 1 & R_o & 1 & R_o
\end{array}$$

where the scaling for W_a/H has been obtained from the mass conservation equation (5). Therefore the Q-G thermodynamic equation is given by

$$\boxed{\frac{D\theta'}{Dt} + w_a\frac{\partial\theta_o}{\partial z^*} = 0} \tag{12}$$

i.e. the potential temperature will not be constant following the geostrophic velocity since, although the ageostrophic vertical velocity is small, the vertical stratification is strong and so the term involving vertical advection by the agoestrophic velocity is just as important as the horizontal advection by the geostrophic velocities in the overall conservation of potential temperature following the fluid parcel.

The Q-G primitive equations may then be summarised as follows

$$\frac{Du_g}{Dt} - f_o v_a - \beta y v_g = 0 \tag{13}$$

$$\frac{Dv_g}{Dt} + f_o u_a + \beta y u_g = 0 \tag{14}$$

$$\frac{D\theta'}{Dt} + w_a\frac{\partial\theta_o}{\partial z^*} = 0 \tag{15}$$

$$\left.\frac{\partial u_a}{\partial x}\right|_{z^*} + \left.\frac{\partial v_a}{\partial y}\right|_{z^*} + \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o w_a^*\right) = 0 \tag{16}$$

$$\frac{\partial\Phi}{\partial z^*} = \frac{RT}{H}. \tag{17}$$

7.4 The Q-G PV equation

The vorticity equation can be formed from 13 and 14 as for the shallow water system (see Section 5.4)

$$\frac{D_g\zeta_g}{Dt} + \beta v_g + f_o\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) = 0$$

where ζ_g is the geostrophic vorticity ($\partial v_g/\partial x - \partial u_g/\partial y$ and D_g/Dt is the material derivative following the geostrophic flow. The continuity equation 16 can then be used to re-write this as

$$\frac{D_g\zeta_g}{Dt} + \beta v_g - \frac{f_o}{\rho_o}\frac{\partial}{\partial z^*}(\rho_o w_a^*) = 0$$

The w_a^* can then be written in terms of θ from the thermodynamic equation (7)

$$\frac{D_g\zeta_g}{Dt} + \beta v_g + \frac{f_o}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o\frac{D_g(\theta-\theta_o)}{Dt}\frac{1}{d\theta_o/dz^*}\right)$$

ρ_o is the reference density profile which depends on z^* only. So, this third term can be manipulated to be in a more convenient form

$$\frac{f_o}{\rho_o}\frac{\partial}{\partial z^*}\left(\frac{D_g}{Dt}\left(\rho_o\frac{\theta-\theta_o}{d\theta_o/dz^*}\right)\right) = \frac{f_o}{\rho_o}\left[\frac{D_g}{Dt}\left(\frac{\partial}{\partial z^*}\left(\rho_o\frac{\theta-\theta_o}{d\theta_o/dz^*}\right)\right) + \frac{\rho_o}{d\theta_o/dz}\left(\frac{\partial u_g}{\partial z^*}\frac{\partial\theta}{\partial x} + \frac{\partial v_g}{\partial z}\frac{\partial\theta}{\partial y}\right)\right]$$

But from thermal wind balance (Equation 50 in Section 2) it can be demonstrated that the the two terms in the second quantity will cancel each other out to leave

$$\frac{D_g\zeta_g}{Dt} + \beta v_g + \frac{f_o}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o\frac{\theta-\theta_o}{d\theta_o/dz^*}\right)$$

The term βv_g can be written as $D_g f/Dt$ and so we are left with the PV conservation equation

$$\frac{D_g q}{Dt} = 0 \qquad q = f(y) + \zeta_g + \frac{f_o}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o\frac{\theta-\theta_o}{d\theta_o/dz^*}\right) \tag{18}$$

Finally, this equation is more useful if it is written entirely in terms of the geostrophic stream function. The geostrophic stream function can be defined as before

$$\psi' = \frac{\phi'}{f_o}$$

where the geopotential has been divided into the reference profile that would occur if the atmosphere was isothermal and the perturbation component

$$\phi = \Phi_o(z^*) + \phi'(x, y, z^*, t) \tag{19}$$

Note that since the background profile is the profile that would occur in an isothermal atmosphere and since log pressure height is identically equal to height in that case then

$$\frac{d\Phi_o}{dz^*} = g \tag{20}$$

The buoyancy frequency associated with the vertical stratification is given by

$$N^2 = \frac{g}{\theta_o}\frac{d\theta_o}{dz^*}$$

The third term in 22 may therefore be written as

$$\frac{f_o}{\rho_o}\frac{\partial}{\partial z^*}\left(\frac{\rho_o g}{N^2}\frac{\theta'}{\theta_o}\right) \tag{21}$$

Then, some rearranging of the relationship between potential temperature and temperature gives

$$\theta' = T'\left(\frac{p}{p_o}\right)^{-\kappa} \rightarrow ln\theta' = lnT' - \kappa ln\left(\frac{p}{p_o}\right) = lnT' + \frac{\kappa z^*}{H} \rightarrow \theta' = T' exp\left(\frac{\kappa z^*}{H}\right)$$

Then the hypsometric equation (17) can be used to write potential temperature in terms of geopotential height

$$\theta' = \frac{H}{R}\frac{\partial \phi'}{\partial z^*} exp\left(\frac{\kappa z^*}{H}\right)$$

Similarly

$$\theta_o = \frac{H}{R}\frac{\partial \Phi_o}{\partial z^*} exp\left(\frac{\kappa z^*}{H}\right)$$

Plugging these into 21 and remembering 20 this third term may be written in terms of geopotential height (ϕ') and therefore in terms of the stream function (ψ') to give the final form of the PV conservation equation

$$\boxed{\frac{D_g q}{Dt} = 0 \qquad q = f(y) + \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o \frac{f_o^2}{N^2}\frac{\partial}{\partial z^*}\right)\right]\psi'} \tag{22}$$

7.5 Vertically propagating Rossby waves in a stratified atmosphere.

The Q-G PV equation (22) can be used to solve for the wave solutions in a stratified atmosphere. The climatological wind in the atmosphere is predominantely in the zonal direction with variations both horizontally and vertically. We therefore consider small perturbations to a background zonal state that varies in the y and z^* directions i.e. $u = u_o + u'$, $v = v'$ and $q = q_o + q'$ where dashed quantities are small compared to the basic state. Linearising PV conservation and noting that $\partial q_o/\partial x = 0$ and $\partial q_o/\partial t = 0$ gives

$$\frac{\partial q'}{\partial t} + u_o\frac{\partial q'}{\partial x} + v'\frac{\partial q_o}{\partial y} = 0 \tag{23}$$

where q' is given by (22) so that the equation that must be satistifed by the perturbation stream function (Ψ') is

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right)\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_o}\frac{\partial}{\partial z^*}\left(\rho_o\frac{f_o^2}{N^2}\frac{\partial}{\partial z^*}\right)\right]\psi' + q_{oy}\frac{\partial \Psi'}{\partial x} = 0 \tag{24}$$

Consider a wave solution of the form $\Psi'(x, y, z^*, t) = exp\left(\frac{z^*}{2H}\right)\psi(x, y, z, t)$. The factor $exp\left(\frac{z^*}{2H}\right)$ simplifies matters as there are non constant coefficients in the equation. The amplitude of the disturbance would be expected to grow exponentially with height since density falls exponentially with height.

The stratification N^2 is a constant since it is the stratification of the background state which has a potential temperature distribution $\theta_o \propto exp\left(\kappa z/H\right)$ so that $\frac{1}{\theta_o}\frac{d\theta_o}{dz}$ = constant. The term involving the vertical derivative of the stream function in 24 can therefore be re-arranged to give

$$\frac{f_o^2}{N^2} exp\left(\frac{z}{2H}\right)\left[\frac{\partial^2\psi'}{\partial z^2} - \frac{1}{4H^2}\psi'\right]$$

So, this, together with plugging the wave solution into the other terms gives

$$\left(\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right)\left[\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} + \frac{f_o^2}{N^2}\left[\frac{\partial^2\psi'}{\partial z^2} - \frac{1}{4H^2}\psi'\right]\right] + q_{oy}\frac{\partial\psi'}{\partial x} = 0 \qquad (25)$$

Consider a solution of the form $\psi' = \psi_o exp\left[i(kx + ly + mz - \omega t)\right]$ where m is the vertical wavenumber. Here the WKB assumption has been made, that is that the wavenumbers are slowly varying functions of latitude and height. The variations of the wave are on a smaller scale than the variations of the mean flow.

This gives the following dispersion relation for vertically propagating Rossby waves in a stratified atmosphere

$$\boxed{\omega = u_o k - \frac{q_{oy}k}{\left(k^2 + l^2 + \frac{f_o^2}{N^2}m^2 + \frac{f_o^2}{4N^2H^2}\right)}} \qquad (26)$$

There are several interesting things to note from this dispersion relation.

- It can be compared with the two-layer dispersion relations (Eq. (25) and (26) in section 6). The differences are that now we have a zonal wind profile that varies with y and z ($\overline{u}(y,z)$) so the background gradient of potential vorticity is not just due to the gradient of planetary vorticity but is also due to the background gradient of potential vorticity associated with the zonal wind structure. The basic state zonal wind profile can be written $\overline{\Psi} = -Uy$ and putting that into 22, the expression for the background gradient of potential vorticity is

  $$\overline{q}_{oy} = \beta + \frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o\frac{f_o^2}{N^2}\frac{\partial u}{\partial z}\right)$$

  So, the basic state PV gradient depends on the planetary vorticity gradient, the meridional curvature of the zonal wind and the vertical shear and curvature of the zonal wind.

  The fully stratified dispersion relation can be seen to be comparable to the two layer dispersion relations. If the vertical wavenumber m =0 then the dispersion relation is very similar to the dispersion relation for the barotropic stream function in the 2-layer system. There is an extra constant ($f_o^2/4N^2H^2$), which comes in from the fact that density depends on height. Similarly the $m = 1$ case could be compared with the baroclinic stream function dispersion relation in the two-layer model. This would be a baroclinic mode that has a slower phase speed than the barotropic mode. The difference now is that we don't have discrete layers and there is no boundary at the top, so the wavenumber doesn't have to be quantised and could be anything. There are a whole spectrum of waves that can exist with different vertical structures that will have different phase speeds.

- The zonal phase speed can be found from the dispersion relation. It is given by

$$c_{px} = \frac{\omega}{k} = U_o - \frac{q_{oy}}{\left(k^2 + l^2 + \frac{f_o^2}{N^2}m^2 + \frac{f_o^2}{4N^2H^2}\right)} \tag{27}$$

Therefore, much like the Rossby waves discussed in the shallow water system, these waves have a westward phase speed relative to the background flow.

- Equation 27 can be rearranged to give the following expression for the vertical wavenumber.

$$m^2 = \frac{N^2}{f_o^2}\left[\frac{\overline{q}_o y}{U_o - c_{px}} - (k^2 + l^2)\right] - \frac{1}{4H^2} \tag{28}$$

Since the perturbation stream function amplitude is proportional to $exp(imz^*)$, if m is real then the wave is vertically propagating whereas if m is complex the wave is vertically evanescent.

So, for vertical propagation of the waves the following condition must be satisfied

$$0 < U_o - c_{px} < \frac{q_{oy}}{(k^2 + l^2) + \frac{f_o^2}{4N^2H^2}} = U_c \tag{29}$$

That is, for vertical propagation the zonal wind must be greater than the phase speed, but less than some critical value given by $U_c + c_{px}$. This critical value depends both on the basic state PV gradient as well as the horizontal scale of the waves.

For stationary waves ($c_{px} = 0$) this is known as the **Charney-Drazin Criterion**.

This has important consequences, particularly for the stratosphere. Westerly winds exist in the winter stratosphere, but Easterly winds exist in the summer stratosphere. Therefore, vertical propagation of waves into the stratosphere only occurs in the winter hemisphere. Moreover, since the wind speed has to be less than a critical value which depends on the horizontal scale, it can be seen that this critical value will be smaller for higher wavenumbers (smaller scale waves). Therefore, in regions of strong westerly winds such as the high latitude winter stratosphere, only the smallest wavenumbers (largest horizontal scales) may propagate vertically. This explains the lack of high wavenumber disturbances in the stratosphere and their restriction to the troposphere.

- An expression for the vertical group velocity can by found from 27

$$c_{gz} = \frac{\partial\omega}{\partial m} = \frac{2f_o^2 q_{oy} km}{\left(k^2 + l^2 + \frac{f_o^2}{N^2m^2} + \frac{f_o^2}{4N^2H^2}\right)^2} \tag{30}$$

All the terms in this expression are positive so the vertical group velocity is positive and the wave propagates information upward.

- Lines of constant phase $(kx + ly + mz - \omega t)$ slope westward with height.

- Vertically propagating waves transport heat poleward. From the hypsometric equation and the relation ship between the stream function and geopotential ($\psi' = \phi'/f_o$), the temperature perturbation associated with the wave can be written as

$$T' = \frac{Hf_o}{R}\frac{\partial\psi'}{\partial z}$$

Also,

$$v' = \frac{\partial \psi'}{\partial x}$$

From these, an expression for the zonal mean poleward heat flux ($\overline{v'T'}$) associated with the wave can be found as follows

$$v'T' = ik\psi_o exp\left(\frac{z}{2H}\right) exp(i\chi) \times \frac{Hf_o}{R}\psi_o exp\left(\frac{z}{2H}\right)\left(\frac{1}{2H} + im\right) exp(i\chi)$$

denoting the phase $(lx + ly + mz)$ by χ. This gives

$$v'T' = \frac{kHf_o\psi_o^2}{R} exp\left(\frac{z}{H}\right)\left[i(cos\chi + isin\chi)\left[\frac{1}{2H}cos\chi + imcos\chi + \frac{1}{2H}isin\chi - msin\chi\right]\right]$$

Re-arranging and taking only the real component gives

$$v'T' = \frac{kHf_o\psi_o^2}{R} exp\left(\frac{z}{H}\right)\left(-a(cos^2\chi - sin^2\chi)m - \frac{1}{H}sin\chi cos\chi\right)$$

Using the trigonometric identity $cos^2\chi - sin^2\chi = 2cos^2\chi - 1$ this can be written

$$v'T' = \frac{kHf_o\psi_o^2}{R} exp\left(\frac{z}{H}\right)\left(m - 2mcos^2\chi - \frac{1}{H}sin\chi cos\chi\right)$$

Upon taking the zonal mean all expressions involving $sin\chi$ or $cos\chi$ will go to zero and we are left with

$$\overline{v'T'} = \frac{kHf_o\psi_o^2}{R} exp\left(\frac{z}{H}\right) m$$

i.e. the zonal mean poleward heat transport is positive. So, any wave that is vertically propagating will have a westward phase tilt with height and will be associated with a positive poleward heat flux.