2 The equations governing atmospheric flow.

2.1 Introduction to Lagrangian (Material) derivatives

The equations governing large scale atmospheric motion will be derived from a Lagrangian perspective i.e. from the perspective in a reference frame moving with the fluid. The Eulerian viewpoint is normally the one we are concerned with i.e. what is the temperature or wind at a fixed point on the Earth as a function of time. But, the Lagrangian perspective is useful for deriving the equations and understanding why the Eulerian winds and temperatures evolve in the way they do.

The **material derivative** (D/Dt) is the rate of change of a field following the air parcel. For example, the material derivative of temperature is given by

$$\boxed{\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T,} \tag{1}$$

where the first term on the RHS is the Eulerian derivative (i.e. the rate of a change at a fixed point) and the second term is the advection term i.e. the rate of change associated with the movement of the fluid through the background temperature field. If the material derivative is zero then the field is conserved following the motion and the local rate of change is entirely due to advection.

2.1.1 Material derivatives of line elements

When deriving the equations of motion from a lagrangian perspective we will consider fluid elements of fixed mass. But, their volume may change and it is therefore necessary to consider the material derivative of line elements (vectors joining two points). Consider Fig. 1 which shows a line element $\delta\vec{r}$ at position $\vec{r}$ at a time t. This line element could
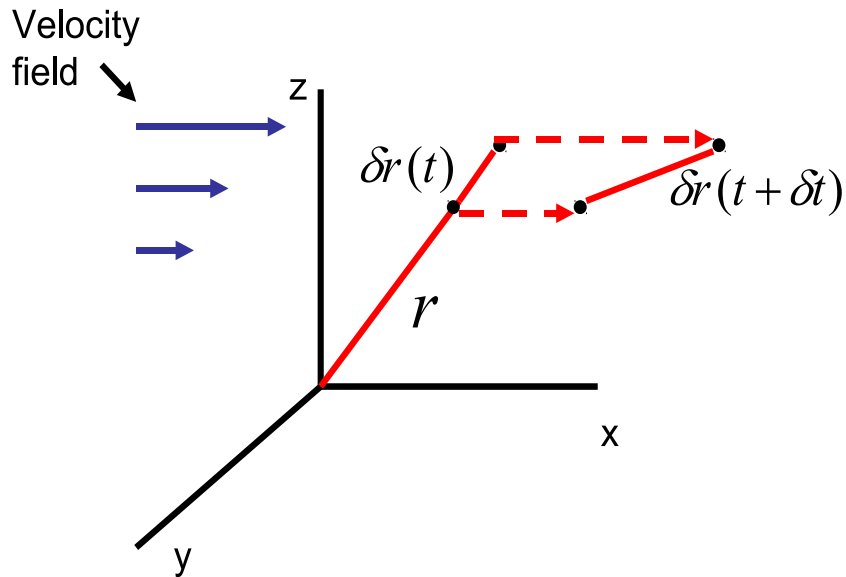

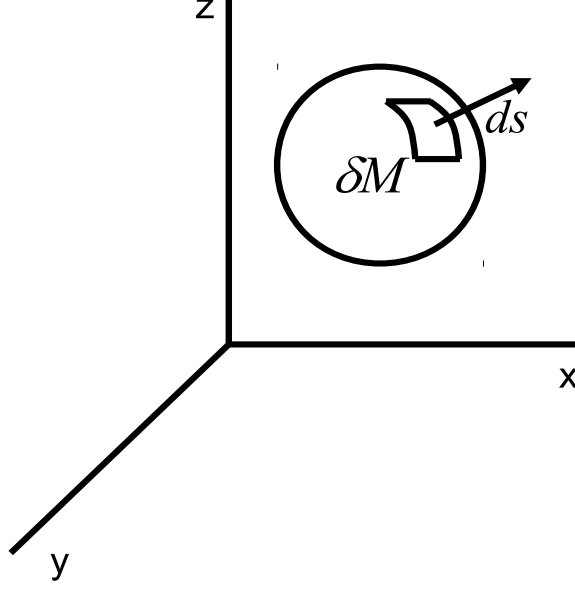**Figure 1:** Schematic of a line element moving with a velocity field.

Figure 2: The infinitesimal mass element δM.

represent, for example, the side of an infinitesimal fluid element. If the line element were connecting two fixed points in space then it will not change with time, but, if it connects two points that are moving with the velocity field of the fluid it will change its position and orientation over time. The line element at time $t + \delta t$ is given by

$$\delta \vec{r}(t+\delta t) = \delta \vec{r}(t) + \vec{v}(\vec{r}+\delta \vec{r})\delta t - \vec{v}(\vec{r})\delta t \rightarrow \frac{\delta \vec{r}(t + \delta t) - \delta \vec{r}(t)}{\delta t} = \vec{v}(\vec{r}+\delta \vec{r}) - \vec{v}(\vec{r}) = \delta \vec{r}\frac{\partial \vec{v}}{\partial \vec{r}} \quad (2)$$

The left hand side is just the material derivative of the line element. So we have the important result that

$$\boxed{\frac{D\delta \vec{r}}{Dt} = \delta \vec{r} \cdot \nabla \vec{v}} \quad (3)$$

where $\vec{v} = \frac{D\vec{r}}{Dt}$. So, different shear terms can affect the size and orientation of line elements. For a particular dimension q we have the general result that

$$\frac{1}{\delta q}\frac{D\delta q}{Dt} = \frac{\partial}{\partial q}\left(\frac{Dq}{Dt}\right) \quad (4)$$

The equations that govern the atmospheric flow will now be derived from a Lagrangian perspective. These are **Mass Conservation, Momentum Conservation** and **Energy Conservation**.

2.2 Conservation of mass (Equation of continuity)

We consider an infinitesimal fluid element of constant mass δM depicted in Fig. 2 whose density (ρ) and volume (δV) may vary with time. Since the mass of the fluid element is constant then $D\delta M/Dt = 0$, where $\delta M = \rho \delta V$. So,

$$\frac{D\delta M}{Dt} = 0 \rightarrow \frac{D\rho}{Dt}\delta V + \frac{D\delta V}{Dt}\rho = 0 \quad (5)$$

Consider first the rate of change of the infinitesimal volume

$$\frac{D\delta V}{Dt} = \frac{D}{Dt}(\delta x \delta y \delta z) = \delta y \delta z \frac{D\delta x}{Dt} + \delta x \delta z \frac{D\delta y}{Dt} + \delta x \delta y \frac{D\delta z}{Dt} \tag{6}$$

But, from the material derivative of line elements (Eq. 3) this can be written

$$\frac{D\delta V}{Dt} = \delta x \delta y \delta z \left(\left(\frac{\partial}{\partial x} \frac{D\delta x}{Dt} \right) + \left(\frac{\partial}{\partial y} \frac{D\delta y}{Dt} \right) + \left(\frac{\partial}{\partial z} \frac{D\delta z}{Dz} \right) \right) = \delta V \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = (\nabla.\vec{v})\delta V. \tag{7}$$

So,

$$\frac{D\delta V}{Dt} = (\nabla.\vec{v})\delta V \tag{8}$$

So, the fractional rate of change of the volume is equal to the divergence of the velocity field. So, back to the eqution for mass coninuity we have

$$\frac{D\rho}{Dt}\delta V + \frac{D\delta V}{Dt}\rho = 0 \rightarrow \frac{D\rho}{Dt}\delta V + \rho(\nabla.\vec{v})\delta V = 0 \rightarrow \frac{D\rho}{Dt} + \rho\nabla \cdot \vec{v} = 0 \tag{9}$$

for arbitrary δV. This can also be written in flux form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho) = 0 \tag{10}$$

To summarise, mass continuity is given by

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{v} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho) = 0} \tag{11}$$

Thus, a fluid element in a divergent flow field will experience a reduction in density.

2.3 Conservation of momentum

Here, we are again considering our infinitesimal air parcel of constant mass (δM) with variable volume (δV) and density ρ. We want to know how the momentum of the parcel will change. The rate of change of momentum will be equal to the sum of the forces acting on the parcel.

$$F = \frac{D(\delta M \vec{v})}{Dt} = \delta M \frac{D\vec{v}}{Dt} \tag{12}$$

There are several forces we may have to consider when examining the motion of fluid elements in the atmosphere or ocean:

- **Pressure** - Consider the vector area element whose normal vector is given by $d\vec{s}$ in Fig. 2. The pressure acting on the area element is $d\vec{F_p} = -pd\vec{s}$. Integrating over the surface area of the element gives $\vec{F_p} = -\int pd\vec{s}$ which from the divergence theorem can be written $\int \nabla p dV \approx \nabla p \delta V$ as $\delta V \rightarrow 0$.

- **Gravity** - The gravitational force acting on the fluid element in geometric height coordinates is $F_g = -\delta M g$ which is often written as $F_g = -\delta M \nabla \Phi$ where $\Phi = gz$ is the **Geopotential** and g is the acceleration due to gravity which for many purposes can be taken to be a constant. The reason for writing it in this form is that we may consider flow in different height coordinates (commonly pressure is used). Thus a particular vertical level in our coordinate system may not necessarily be horizontal and there may ba a component of gravity acting along that level.
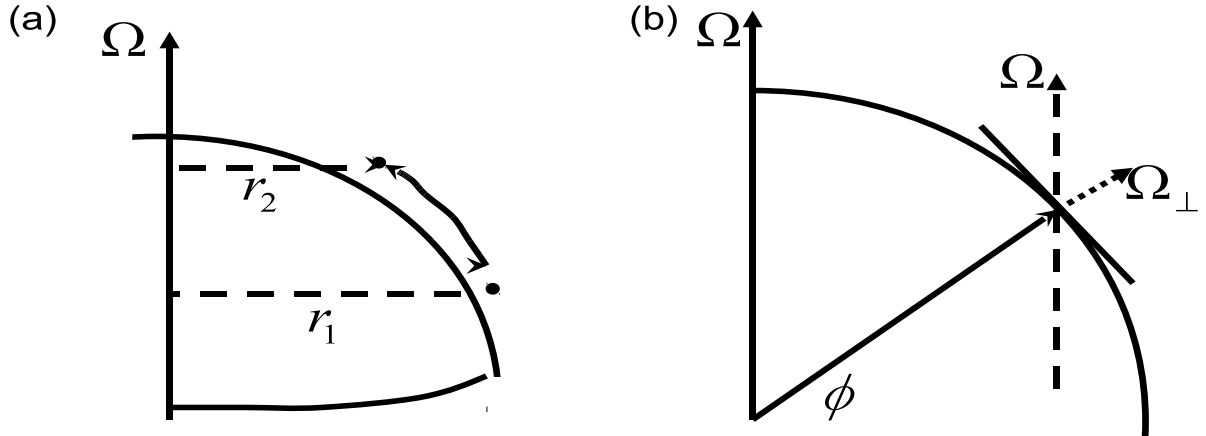
Figure 3: (a) Schematic illustrating the coriolis force as conservation of angular momentum, (b) schematic illustrating the tangent plane approximation.

- **Viscosity** - Viscosity is a measure of the resistance of a fluid. The viscosity arises from shear stress between layers of a fluid that are moving with different velocities. The viscous force per unit mass is often written as $\vec{F}_\nu = \nu \nabla^2 \vec{v}$ where ν is the kinematic viscosity. For the atmosphere below around 100km the viscosity is so small that it is normally negligible except for within a few cm at the Earth's surface where the vertical shear is very large. (See Holton 1.4.3 for more on viscosity). The **Reynolds number** is a measure of the importance of viscosity. It is the ratio of inertial forces to viscosity. In most of the atmosphere the Reynolds number is large and viscosity is relatively unimportant.

So, our rate of change of momentum in Eq. 12 is given by the sum of these forces. Dividing through by δM gives us an expression for the acceleration of the fluid parcel in an inertial reference frame.

$$\frac{D\vec{v}}{Dt} = -\frac{\nabla \vec{p}}{\rho} - \nabla \Phi + \nu \nabla^2 \vec{v} \tag{13}$$

If the fluid is at rest then we have **hydrostatic balance**

$$-\frac{\nabla p}{\rho} - \nabla \Phi = 0 \rightarrow \frac{\partial p}{\partial z}\vec{k} = -\rho g \vec{k} \tag{14}$$

2.3.1   Adding in rotation

In the above section we related the rate of change of momentum of our fluid element to the sum of the forces acting on the fluid element. But, that only applies when we are considering motion in an inertial reference frame. Normally when examining atmospheric motion we are not looking at motion in an inertial reference frame. Rather we are concerned with how the atmosphere is moving with respect to the Earth's surface i.e. in a rotating reference frame. We therefore have to take this into account when determining momentum balance.

For example, consider a parcel of air which is at a fixed position with respect to the Earth's surface. The Earth rotates with an angular velocity $\vec{\Omega}$. The position of the fluid

element does not alter in the rotating reference frame of the Earth. But, from the point of a view of an observed in the Inertial reference frame the position vector will move as it rotates with the Earth. The rate of change of the position vector as viewed from the inertial reference frame is given by (See Holton 2.1.1)

$$\left(\frac{D\vec{r}}{Dt}\right)_I = \vec{\Omega} \times \vec{r} \tag{15}$$

If the air parcel now has a velocity with respect to the fixed surface of the Earth denoted by $(D\vec{r}/Dt)_R$, then an observed in the inertial reference frame will see the air parcel moving with a velocity

$$\left(\frac{D\vec{r}}{Dt}\right)_I = \left(\frac{D\vec{r}}{Dt}\right)_R + \left(\vec{\Omega} \times \vec{r}\right) \tag{16}$$

i.e. the sum of the velocity in the rotating reference frame and the velocity associated with the movement of the reference frame.

$$\vec{v}_I = \vec{v}_r + (\Omega \times \vec{r}) \tag{17}$$

If we now consider the rate of change of the inertial velocity in the inertial reference frame: it will be related to the rate of change of the inertial velocity in the rotating reference frame by the same transformation (16) i.e.

$$\left(\frac{D\vec{v}_I}{Dt}\right)_I = \left(\frac{D\vec{v}_I}{Dt}\right)_R + \vec{\Omega} \times \vec{v}_I. \tag{18}$$

But, combining with (17) it can be shown that

$$\left(\frac{D\vec{v}_I}{Dt}\right)_I = \left(\frac{D\vec{v}_R}{Dt}\right)_R + 2\vec{\Omega} \times \vec{v}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{19}$$

where the 2nd and 3rd terms on the RHS are the Coriolis force and the centrifugal force respectively. These are ficticious forces introduced by the fact that we are examining the motion in a non-inertial reference frame.

- **The centrifugal force** - The can be rewritten as $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}_\perp)$ where $\vec{r}_\perp$ is the component of the position vector that is perpendicular to the axis of rotation. This can can be re-written as $(\vec{\Omega}.\vec{r}_\perp)\vec{\Omega} + (\vec{\Omega}.\vec{\Omega}).\vec{r}_\perp$ where the first term on the RHS is zero giving $\Omega^2 \vec{r}_\perp$. This can then be written as the gradient of a scalar potential $(\nabla \Phi_{CE})$ where $\Phi_{CE} = \Omega^2 \vec{r}_\perp/2$. Therefore, the centrifugal force is often combined with the gravitational force in (13) but for many purposes the centrifugal force is small and can actually be neglected.

- **Coriolis force** - The coriolis force can be thought of in terms of angular momentum conservation. For examine, consider an air parcel starting at position (1) at low latitudes in the NH in Fig. 3 (a). Suppose now a force acts to shift that air parcel poleward to position (2). No torque has acted on the parcel in the zonal direction and so the angular momentum of the air parcel around the Earths axis of rotation must be conserved i.e. $\Omega_1 r_1^2 = \Omega_2 r_2^2$, but $r_1 > r_2 \rightarrow \Omega_1 < \Omega_2$. So, as the air parcel moves poleward, it's angular velocity in the zonal direction increases. So, an observer at position (1) rotating with the Earth would see a trajectory of the air parcel that looks like it is being deflected to the right.

So, in (13) the acceleration of the air parcel is related to the forces that are acting on it. But, that applies for motion in an inertial reference frame. What we normally care about is the motion in a rotating reference frame i.e. $\left(\frac{D\vec{v}_R}{Dt}\right)_R$. We can therefore combine (13) and (19) to give

$$\left(\frac{D\vec{v}_R}{Dt}\right)_R + 2\vec{\Omega} \times \vec{v}_R = \frac{\nabla p}{\rho} - \nabla\Phi + \nu\nabla^2\vec{v} \tag{20}$$

where we have either neglected the coriolid force or lumped it in with the gravitational potential.

From now on we shall drop the subscript R and assume we are examining motion in the rotating reference frame of the Earth. So, to conclude, momentum balance in the rotating reference frame of the Earth where pressure, gravity, viscous and possibly other frictional forces are acting is given by

$$\boxed{\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{\nabla p}{\rho} - \nabla\Phi + \nu\nabla^2\vec{v} \qquad (+friction)} \tag{21}$$

2.3.2 Scale analysis of momentum balance

Considering the typical horizontal velocity (U) and typical length scales (L) of the system we can perform scale analysis on the the first two terms of Eq. 39. The first term

$$\frac{D\vec{v}}{Dt} \sim \frac{U}{T_A} \sim \frac{U^2}{L} \tag{22}$$

where we have made use of the fact that the relevant timescale for the material derivative is the advective timescale. The second term scales as

$$2\Omega \times \vec{v} = 2\Omega\vec{v}sin\phi \sim fU \tag{23}$$

where f is the coriolis parameter. The angle ϕ is the angle between the rotation vector and the velocity vector which when considering the horizontal velocity components on the Earth's surface is simply the latitude (See Fig. 3 (b)). Thus $1./f$ is the relevant timescale for the effects of the Earth's rotation as was discussed in Section 1. Taking the ratio of the first term to the second term we end up with the ratio U/fL which is the Rossby Number R_o. So, the Rossby number is a dimensionless parameter that comes from scale analysis of momentum balance. It is the ratio of the intrinsic acceleration to the acceleration associated with the coriolis force. If the Rossby number is small then the effects of the Earths rotation are relatively important and vice-versa. If we are in the regime of small Rossby number then if you apply a force on an air parcel and you're in the rotating reference frame of the Earth then you're not going to see the parcel accelerating in the direction of the force because there will be an acceleration due to the coriolis force. We see from the terms involved in the Rossby number that the coriolis force is relatively important if we are considering motion that is slow and/or large scale. This is the type of motion that we will be concerned with.

2.4 Conservation of Energy (Thermodynamic Equation)

Again we consider our air parcel of fixed mass δM. We define I=the internal energy per unit mass, η the entropy per unit mass and α the volume per unit mass. Thus the total

internal energy is given by $\delta U = I \delta M$, the total entropy is given by $\delta S = \eta \delta M$ and the total volume is given by $\delta V = \alpha \delta M$. We take the first law of thermodynamics as our starting point for deriving the thermodynamic equation for atmospheric flow. The first law of thermodynamics may be written as

$$dU = TdS - pdV \tag{24}$$

where d represents the change in our thermodynamic variable. That is the change in internal energy of our mass element is given by the difference between the heat input into the mass element and the work done by the mass element. This is in terms of the extensive quantities so it depends on the size of the system. But, we can write it in terms of the quantities per unit mass (or intensive variables) as follows

$$d(I\delta M) = Td(\eta\delta M) - pd(\alpha\delta M) \tag{25}$$

So, over some time interval δt we have

$$\frac{d(I\delta M)}{\delta t} = \frac{Td(\eta\delta M)}{\delta t} - \frac{pd(\alpha\delta M)}{\delta t} \tag{26}$$

As the time interval δt tends to zero these tend to the rate of change following the parcel (the material derivative) so

$$\frac{D(I\delta M)}{Dt} = T\frac{D(\eta\delta M)}{Dt} - p\frac{D(\alpha\delta M)}{Dt} \tag{27}$$

But, δM is conserved following the motion.

$$\frac{DI}{Dt} = T\frac{D\eta}{Dt} - p\frac{D\alpha}{Dt} \tag{28}$$

This is our first law of thermodynamics in terms of the entropy per unit mass and the volume per unit mass. These are rather difficult quantities to measure, but matters may be simplified when we are considering motion of an ideal gas, which we are for most purposes in atmospheric dynamics. It is much easier if we can use pressure (p) and temperature (T) as our thermodynamic variables as these are quantities we can measure. For an ideal gas we have several useful relationships

$$p\alpha = RT \qquad I = C_V T \qquad C_p - C_V = R \tag{29}$$

where R is the gas constant, C_V is the specific heat capacity at constant volume and C_P is the specific heat capacity at constant pressure. So, we can rewrite 28 as

$$C_V\frac{DT}{Dt} = T\frac{D\eta}{Dt} - R\frac{DT}{Dt} + \frac{RT}{p}\frac{Dp}{Dt} \tag{30}$$

which after some rearranging can be written

$$T\frac{D\eta}{Dt} = C_p\frac{DT}{Dt} - \frac{RT}{p}\frac{Dp}{Dt} \tag{31}$$

This is an expression for the rate of change of entropy as a function of temperature and pressure. But, if we are considering adiabatic motion of an ideal gas then entropy should

be conserved. This leads us to the definition of potential temperature. For adiabatic motion of an ideal gas we have

$$\frac{D\eta}{Dt} = \frac{C_p}{T}\frac{DT}{Dt} - \frac{R}{p}\frac{Dp}{Dt} = C_p\left(\frac{DlnT}{Dt} - \frac{Dlnp^\kappa}{Dt}\right) = C_P\frac{Dln\frac{T}{p^\kappa}}{Dt} = 0 \qquad (32)$$

where $\kappa = R/C_p$. Therefore this ratio T/p^κ must be a constant since the C_p is a constant and non-zero. If we have an air parcel that has a temperature T at a pressure p then if it is brought adiabatically to a reference pressure p_o then it will have a temperature (θ) at that reference pressure given by

$$\frac{\theta}{p_o^\kappa} = \frac{T}{p^\kappa} \rightarrow \theta = T\left(\frac{p_o}{p}\right)^\kappa \qquad (33)$$

This quantity θ is known as the potential temperature. It is the temperature that an air parcel would have if it were brought adiabatically to the reference pressure p_o. The reference pressure is normally taken to be the surface pressure ($\sim$1000hPa). Potential temperature increase upward in the troposphere even though temperature decreases because the pressure is also decreasing. A positive potential temperature gradient implied stability as given by the Brunt-Vaisala frequency in section 1.

By comparison of 33 and 32 it can be seen that the rate of change of entropy is related to the potential temperature by

$$\frac{D\eta}{Dt} = C_p\frac{Dln\theta}{Dt} \qquad (34)$$

and thus

$$\eta = C_p ln\theta \qquad (35)$$

Surface of constant potential temperature are therefore surface of constant entropy (isentropic surfaces). We can see from Eqs. 32 and 33 that potential temperature must be conserved following adiabatic motion of an ideal gas. We can therefore write our thermodynamic equation as

$$\frac{D\theta}{Dt} = 0 \qquad (36)$$

If there are sources of diabatic heating then we can make use of 34 and the relationship between the heat input and the change in entropy ($dQ = Td\eta$) to write the thermodynamic equation in the presence of diabatic effects as

$$C_p\frac{D\theta}{Dt} = \frac{\theta}{T}\dot{Q} \qquad (37)$$

Note: the above derivation of the thermodynamic equation relied on relationships that are valid for an ideal gas. In the atmosphere what we are really concerned with is the motion of an ideal gas although there may need to be modification to include effects like moisture (see Vallis chapter 1). For, liquids a different equation of state is necessary and the thermodynamic equation takes a different form.

2.5   Summary of the primitive equations

The above equations that we have derived for atmospheric flow are often known as the *primitive equations*. The primative equations are summarised as follows

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\vec{v}\rho) = 0 \qquad \text{CONTINUITY} \qquad (38)$$

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$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{\nabla p}{\rho} - \nabla\Phi - \nu\nabla^2\vec{v} \qquad \text{MOMENTUM} \qquad (39)$$

$$\frac{D\theta}{Dt} = 0 \qquad or \qquad C_p\frac{D\theta}{Dt} = \frac{\theta}{T}\dot{Q} \qquad \text{THERMODYNAMIC} \qquad (40)$$

But, here we have 6 variables $(\rho, u, v, w, p, \theta)$ and only 5 equations. We need the diagnostic equation of state which, for an ideal gas is given by

$$p\alpha = RT \qquad (41)$$

We will be using these equations to look at simple cases in the atmosphere and understand why certain aspects of the large scale circulation behave the way they do.

2.6 Geostrophic Balance

Momentum balance in the absence of viscosity and friction is given by

$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{\nabla p}{\rho} - \nabla\Phi. \qquad (42)$$

The assumptions we make about the characterisctics of large scale extra-tropical motion in the atmosphere allow us to identify the dominant balances for such motions. We consider flow for which the Rossby number is small i.e. the first term in 42 is much smaller than the second. In geometric height coordinates we may neglect the component of gravity acting in the horizontal direction. Thus, in the horizontal we have the dominant balance occuring between the second and third terms (the coriolis force and the horizontal pressure gradient). This gives

$$\boxed{(u_g, v_g) = \frac{1}{\rho f}\left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}\right)} \qquad (43)$$

The above expression gives the **Geostrophic wind**: the horizontal wind for which the coriolis force exactly balances the horizontal pressure gradients. This balance is known as **Geostrophic balance**. Geostrophic balance is often a reasonable approximation for large scale motion away from the equator. Examination of Eq. 43 shows that, in the NH the flow will be clockwise around high pressure (anticyclonic) systems whereas the flow will be anti-clockwise around low pressure (cyclonic) systems. The opposite is true in the SH as the coriolis parameter (f) is of opposite sign.

2.7 Hydrostatic balance

In the velocity was zero it can be seen from Eq. 42 that there would be a balance between the vertical pressure gradient and the gravitational force - **Hydrostatic balance**. But, infact, if we make the assumption that we are examining flow of small Rossby number and small aspect ratio then the supplementary handout demonstrates that even in the presence of a non-zero velocity the dominant balance in the vertical is between the vertical pressure gradient and the gravitational force. So, for large scale flow hydrostatic balance is the dominant balance in the vertical.

$$\boxed{\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{\partial\Phi}{\partial z} \qquad or \qquad \frac{\partial p}{\partial z} = -\rho g} \qquad (44)$$

Eq (44) together with the ideal gas law gives an expression for the geopotential of a pressure level (known as the **Hypsometric equation**)

$$\Phi_p = \Phi_s + \int_p^s RT \, dlnp',$$
(45)

where Φ_p is the geopotetial of a particular pressure level and Φ_s is the geopotential of the surface. The geopotential height (Z) is related to the geopotential via $Z = \Phi/g$ and so Eq. 45 gives an expression for the geopotential height of a particular pressure level. It is related to the integral of temperature below that pressure level and is approximately equal to the geometric height in the troposphere and lower stratosphere.

2.7.1 Using pressure as a vertical coordinate

From hydrostatic balance we have a monotonic relationship between pressure and geometric height and so pressure is an equally valid vertical coordinate. It is often advantageous to formulate the primitive equations in pressure coordinates as we lose the dependence on density. Mass conservation becomes

$$\left.\frac{\partial u}{\partial x}\right|_p + \left.\frac{\partial v}{\partial y}\right|_p + \frac{\partial \omega}{\partial p} = 0$$
(46)

where $\omega = Dp/Dt$ is the pressure velocity. So, for a hydrostatic system in pressure coordinates the velocity field is non-divergent. The horizontal component of inviscid momentum balance in the absence of friction in pressure coordinates becomes

$$\frac{D\vec{v}_H}{Dt} + 2\vec{\Omega} \times \vec{v}_H = -\nabla\Phi$$
(47)

where $\vec{v}_H$ represents only the horizontal velocity (u,v) and our geopotential is now a function of $(x, y, p$ and $t)$ since the geopotential height of a pressure level may vary in the horizontal and with time. Geostrophic balance in pressure coordinates becomes

$$(u_g, v_g) = \left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}, \frac{1}{f}\frac{\partial \Phi}{\partial x} \right).$$
(48)

i.e. the geostrophic velocity in pressure coordinates is related to horizontal variations of the geopotential of the pressure surface. It is clear from 48 that the geostrophic velocity in pressure coordinates is non-divergent and so we may define a stream function for the geostrophic flow such that

$$u = \frac{\partial \psi}{\partial y} \qquad\qquad v = -\frac{\partial \psi}{\partial x}$$
(49)

Therefore the geostrophic stream function is given by $\psi = \Phi/f$ (if we neglact horizontal variations in f, (f-plane approximation)). Consider the flow depicted in Fig. 4, it represents either 1) flow around regions of high and low pressure in geometric coordinates or 2) flow around regions of anomalously high and low geopotential height in pressure coordinates. A schematic of the cross section through the left hand side of 4 is also shown. The high and low pressure systems in the geometric coordinates represent deformation of a pressure contour and therefore the low pressure region corresponds to anomalously low geopotential height of our pressure surface and the high pressure region corresponds to anomalously high geopotential height of our pressure surface.
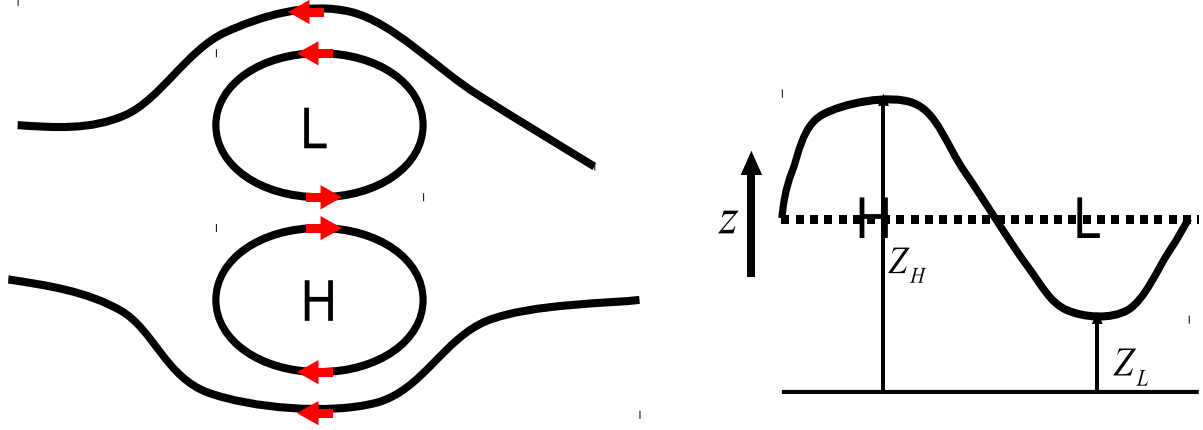
Figure 4: Schematic illustrating the geostrophic flow around high and low pressure systems. This could either represent flow on a geometric height surface around regions of high and low pressure or it could represent flow on a pressure surface around regions of high and low geopotential height as illustrated on the right.

2.8 Thermal wind balance

The combination of geostrophic balance in the horizontal and hydrostatic balance in the vertical provides us with another important balance: **Thermal wind balance**. This relates vertical gradients in the zonal and meridional wind to horizontal gradients in temperature as follows:

$$\frac{\partial u_g}{\partial lnp} = \frac{R}{f}\frac{\partial T}{\partial y} \qquad\qquad \frac{\partial v_g}{\partial lnp} = -\frac{R}{f}\frac{\partial T}{\partial x} \qquad (50)$$

In other words a decrease in temperature with latitude is associated with a positive vertical zonal wind shear and vice-versa (note the above equation is formulated in log(p) coordinates which increases downwards). A decrease in temperature with longitude is associated with a negative vertical gradient of the meridional wind. The relationship between meridional temperature gradients and zonal wind shear can be seen in Fig. 5 which shows the climatology of DJF and JJA zonal wind and temperature. In the troposphere it can be seen that there is a strong negative meridional temperature gradient which is related to the positive vertical wind shear of the sub-tropical and mid-latitude jets. In the winter stratosphere there is also a strong negative meridional temperature gradient associated with the polar night which is related to the positive vertical wind shear of the stratospheric polar vortex.
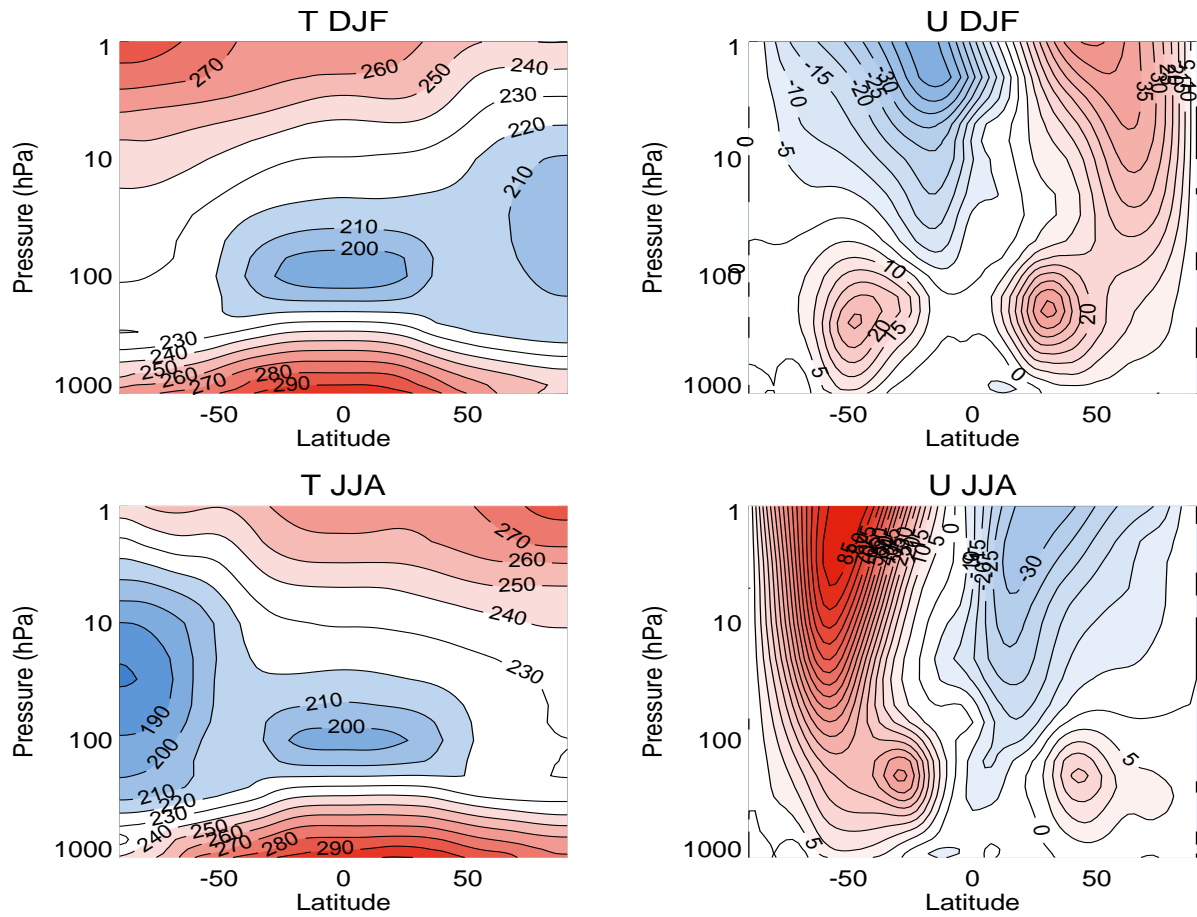
Figure 5: ERA-40 reanalysis data climatologies of (top) DJF (left) Temperature and (Right) Zonal wind and (bottom) corresponding plots for JJA.